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Constructions déterministes pour la régression parcimonieuse

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Avant-propos

Ce mémoire est consacré à la construction et aux performances de designs déterministes pour la sélection de variables et la prédiction d'erreur. Plus précisément, on considère le modèle linéaire

$$y = X\beta^\star + z\,, \tag{I.1}$$

où $y \in \mathbb{R}^n$ est l'observation, $X \in \mathbb{R}^{n\times p}$ est la matrice de design, $\beta^\star \in \mathbb{R}^p$ est la cible, et $z \in \mathbb{R}^n$ est le bruit. On se place dans le cas où $n \ll p$, i.e. le système d'équations représenté par X est fortement sous-déterminé. Notre problématique est de voir si l'on peut retrouver les s plus grands (en valeur absolue) coefficients de $\beta^\star$ à partir des n variables observées avec si possible $n \approx s$, à un facteur log près. Cette question est fondamentale en compression de données où l'on ne garde que les plus grands coefficients pour représenter un vecteurs de grande dimension, mais aussi en génétique où l'on veut trouver les locations du génome qui influencent le plus un trait particulier (e.g. le taux de cholestérol), on encore en imagerie médicale où l'on cherche à reconstruire des images IRM fortement sous-échantillonnées... Dans le cas où le bruit est nul, la meilleure approche (théorique) est alors de considérer l'estimateur combinatoire défini par

$$\beta^c \in \arg\min_{X\beta=y} \|\beta\|_{\ell_0}\,, \tag{I.10}$$

où $\|\,.\,\|_{\ell_0}$ est la « norme » de comptage. Malheureusement, ce programme est NP-difficile et, dans la pratique, il est impossible de l'utiliser pour de valeurs de p grandes. Pour palier à ce problème on étudie une version relaxée de cet estimateur, on parlera alors de relaxation convexe du problème combinatoire. Cette question a été intensivement étudiée au cours des dix dernières années à travers le problème du *Compressed Sensing* [Don06]. En particulier, ce dernier est basée sur l'idée que les vecteurs ayant beaucoup de coefficients nuls, appelés vecteurs parcimonieux ou encore vecteurs creux, appartiennent à un sous-ensemble de la sphère unité ℓ_1 où la norme ℓ_1 n'est pas différentiable. Cette particularité est exploitée grâce aux deux estimateurs suivants : la pénalité ℓ_1 force la solution à être parcimonieuse.

✧ Le lasso [Tib96] :

$$\beta^\ell \in \arg\min_{\beta\in\mathbb{R}^p} \left\{\frac{1}{2}\|y - X\beta\|_{\ell_2}^2 + \lambda_\ell\,\|\beta\|_{\ell_1}\right\}, \tag{I.22}$$

où $\lambda_\ell > 0$ est un paramètre à régler.

✧ Le sélecteur Dantzig [CT07] :

$$\beta^d \in \arg\min_{\beta\in\mathbb{R}^p} \|\beta\|_{\ell_1} \ \text{ sachant que } \ \|X^\top(y - X\beta)\|_{\ell_\infty} \leq \lambda_d\,, \tag{I.23}$$

où $\lambda_d > 0$ est un paramètre à régler.

Il est connu que ces deux programmes peuvent être efficacement résolus à l'aide d'un ordinateur, voir le chapitre I dans lequel on présente les grandes lignes de ces estimateurs. En particulier, on rappelle la définition d'une inégalité oracle (voir Définition I.4 à la page 17), et l'on montre que le sélecteur Dantzig en satisfait une sous la condition RIP, voir ci-dessous pour la définition de cette propriété.

1. La propriété universelle de distorsion

De nombreuses conditions ont été introduites au cours de la dernière décennie. Toutes s'intéressant soit à la consistance ℓ_1, i.e. estimer $\|\beta^l - \beta^\star\|_{\ell_1}$, soit à la prédiction d'erreur, i.e. estimer $\|X\beta^l - X\beta^\star\|_{\ell_1}$, soit encore à l'estimation du support des s-plus grands coefficients. Nous étudions les deux premiers problèmes. En particulier, on considère les conditions suivantes.

- **La propriété d'isométrie restreinte [CRT06b]:** Une matrice $X \in \mathbb{R}^{n\times p}$ satisfait la propriété d'isométrie restreinte d'ordre k si et seulement si il existe $0 < \theta_k < 1$ (le plus petit possible) tel que

$$\forall \gamma \in \mathbb{R}^p \text{ tel que } \|\gamma\|_{\ell_0} \leq k\,, \quad (1-\theta_k)\|\gamma\|_{\ell_2}^2 \leq \|X\gamma\|_{\ell_2}^2 \leq (1+\theta_k)\|\gamma\|_{\ell_2}^2. \qquad \text{(I.15)}$$

  On note cette propriété $\mathrm{RIP}(k,\theta_k)$.

- **La condition de valeur propre restreinte [BRT09]:** Une matrice $X \in \mathbb{R}^{n\times p}$ satisfait $RE(S,c_0)$ si et seulement si

$$\kappa(S,c_0) = \min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\}\\ |\mathcal{S}|\leq S}} \ \min_{\substack{\gamma\neq 0\\ \|\gamma_{\mathcal{S}^c}\|_{\ell_1}\leq c_0\|\gamma_{\mathcal{S}}\|_{\ell_1}}} \frac{\|X\gamma\|_{\ell_2}}{\|\gamma_{\mathcal{S}}\|_{\ell_2}} > 0\,.$$

  La constante $\kappa(S,c_0)$ est appelée la valeur propre (S,c_0)-restreinte.

- **La condition de compatibilité [BvdG09]:** Une matrice $X \in \mathbb{R}^{n\times p}$ satisfait la condition $Compatibility(S,c_0)$ si et seulement si

$$\phi(S,c_0) = \min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\}\\ |\mathcal{S}|\leq S}} \ \min_{\substack{\gamma\neq 0\\ \|\gamma_{\mathcal{S}^c}\|_{\ell_1}\leq c_0\|\gamma_{\mathcal{S}}\|_{\ell_1}}} \frac{\sqrt{|\mathcal{S}|}\,\|X\gamma\|_{\ell_2}}{\|\gamma_{\mathcal{S}}\|_{\ell_1}} > 0\,.$$

  La constante $\phi(S,c_0)$ est appelée la ℓ_1-valeur propre (S,c_0)-restreinte.

- **La condition $\mathbf{H}_{S,1}$ [JN11]:** Une matrice $X \in \mathbb{R}^{n\times p}$ satisfait la condition $\mathbf{H}_{S,1}(\kappa)$ avec $\kappa < 1/2$ si et seulement si pour tout $\gamma \in \mathbb{R}^p$ et pour tout $\mathcal{S} \subseteq \{1,\ldots,p\}$ tel que $|\mathcal{S}| \leq S$, on a

$$\|\gamma_{\mathcal{S}}\|_{\ell_1} \leq \hat\lambda\, S\, \|X\gamma\|_{\ell_2} + \kappa\|\gamma\|_{\ell_1},$$

  où $\hat\lambda$ est le maximum de la norme ℓ_2 des colonnes de X.

Parallèlement aux travaux de A. Juditsky et A. Nemirovski, on a introduit une « nouvelle » condition, la « propriété universelle de distorsion » UDP (voir [dC10] pour une première utilisation de cette propriété avec des graphes expanseurs et [dC11b] pour une approche plus générale). Bien que similaires, les propriété UDP et $\mathbf{H}_{S,1}$ s'appliquent dans des situations différentes. On verra en outre que la propriété UDP est satisfaite par toutes les matrices de plein rang et qu'elle est un lien direct entre la géométrie du noyau du design et les performances du lasso et du sélecteur Dantzig.

Définition 1.1 ([dC11b], $\mathrm{UDP}(S_0, \kappa_0, \Delta)$) — *Une matrice $X \in \mathbb{R}^{n\times p}$ satisfait la propriété universelle de distorsion d'ordre S_0, magnitude κ_0 et paramètre Δ si et seulement si*

- ✦ $1 \leq S_0 \leq p$,
- ✦ $0 < \kappa_0 < 1/2$,
- ✦ *et pour tout $\gamma \in \mathbb{R}^p$, pour tout entier $s \in \{1, \dots, S_0\}$, pour tout sous-ensemble $\mathcal{S} \subseteq \{1, \dots, p\}$ tel que $|\mathcal{S}| = s$, on a*

$$\|\gamma_{\mathcal{S}}\|_{\ell_1} \leq \Delta\sqrt{s}\,\|X\gamma\|_{\ell_2} + \kappa_0\|\gamma\|_{\ell_1}. \tag{II.6}$$

Intuitivement, l'ordre S_0 représente le nombre de coefficients que l'on peut retrouver, on l'appelle le niveau de parcimonie, alors que le paramètre Δ donne la précision des inégalités oracles : plus il est petit plus l'estimation sera bonne. On montre que cette condition est la plus faible parmi toutes les conditions portant sur le lasso et le sélecteur Dantzig (mis à part la condition $\mathbf{H}_{S,1}$, une discussion compare les deux conditions à la section 1.2 page 39).

Proposition 1.1 ([dC11b]) — *Soit $X \in \mathbb{R}^{n\times p}$ une matrice de plein rang, alors les affirmations suivantes sont vraies :*

- ✦ *La condition* $\mathrm{RIP}(5S, \theta_{5S})$ *avec* $\theta_{5S} < \sqrt{2} - 1$ *implique* $\mathrm{UDP}(S, \kappa_0, \Delta)$ *pour toute paire* (κ_0, Δ) *telle que*

$$\left[1 + 2\left[\frac{1-\theta_{5S}}{1+\theta_{5S}}\right]^{1/2}\right]^{-1} < \kappa_0 < 0.5 \quad \text{and} \quad \Delta \geq \left[\sqrt{1-\theta_{5S}} + \frac{\kappa_0 - 1}{2\kappa_0}\sqrt{1+\theta_{5S}}\right]^{-1}. \tag{II.21}$$

- ✦ *La condition $RE(S, c_0)$ implique* $\mathrm{UDP}(S, c_0, \kappa(S, c_0)^{-1})$.
- ✦ *La condition $Compatibility(S, c_0)$ implique* $\mathrm{UDP}(S, c_0, \phi(S, c_0)^{-1})$.

On a ainsi exhibé la condition la plus faible pour obtenir des inégalités oracles avec le lasso et le sélecteur Dantzig. Mais au delà, la propriété UDP est intimement liée à la *distorsion* du noyau du design. Cette dernière mesure l'écart entre une boule euclidienne et l'intersection du noyau avec la boule unité ℓ_1. Un autre point de vue est de dire que la distorsion contrôle le rapport entre la norme ℓ_1 et la norme ℓ_2 sur le noyau du design. De ce point de vue, la condition UDP ne peut pas être généralisée à d'autres normes que les normes ℓ_1 et ℓ_2. Contrairement aux conditions RIP, REC, et $\mathbb{H}_{S,1}$ dont l'analyse s'étend aux normes p pour $1 \leq p \leq \infty$.

Définition 1.2 — *Un sous-espace $\Gamma \subset \mathbb{R}^p$ a une distorsion $1 \leq \delta \leq \sqrt{p}$ si et seulement si*

$$\forall x \in \Gamma, \quad \|x\|_{\ell_1} \leq \sqrt{p}\,\|x\|_{\ell_2} \leq \delta\|x\|_{\ell_1}.$$

Un problème de longue date est de trouver des sections « presque-Euclidiennes » de la boule ℓ_1, i.e. des sous-espaces de faibles codimensions ayant une distorsion la plus petite possible. Il a été établi [Kas77] qu'il existait des sous-espaces Γ_n de codimension n tels que

$$\delta \leq C\left(\frac{p(1+\log(p/n))}{n}\right)^{1/2}, \tag{II.5}$$

où $C > 0$ est une constante universelle. Plus récemment, il a été donné des constructions déterministes approchant cette borne. Une synthèse des principaux résultats se trouve à la section 4 page 30. Les résultats les plus probants sont liés à la théorie des codes correcteurs. Ainsi, la construction de [Ind07] est basée sur l'amplification de la distance minimale d'un code à l'aide d'expanseurs. Alors que la construction [GLR08] est basée sur les codes Low-Density Parity Check (LDPC). Enfin, celle de [IS10] s'inspire du produit tensoriel des codes correcteurs. Il n'est pas si surprenant de voir apparaitre les codes correcteurs dans notre analyse. En effet, ceux-ci peuvent être vus comme le dual du Compressed Sensing : les matrices *Parity-Check* des codes sont de bonnes matrices de design, voir la section 4.2 page 30.

Lemme 1.2 ([dC11b]) — *Soit $X \in \mathbb{R}^{n\times p}$ une matrice de plein rang. On note δ la distorsion de son noyau et ρ_n sa plus petite valeur singulière. Soit $0 < \kappa_0 < 1/2$ alors X satisfait* $\mathrm{UDP}(S_0, \kappa_0, \Delta)$ *où*

$$S_0 = \Big(\frac{\kappa_0}{\delta}\Big)^2 p \quad \text{and} \quad \Delta = \frac{2\delta}{\rho_n}\,.$$

Il est connu [CDD09] que le nombre S_{opt} maximal de coefficients que l'on peut retrouver à partir de n observations satisfait :

$$S_{opt} \approx n/\log(p/n)\,,$$

à une constante multiplicative près. Dans le cas où (II.5) est vrai, le niveau de parcimonie vérifie

$$S_0 \approx \kappa_0^2\, S_{opt}. \tag{II.8}$$

Ainsi, la propriété UDP est satisfaite par toutes les matrices de plein rang à un niveau de parcimonie optimal. On a les inégalités oracles suivantes.

Théorème 1.3 ([dC11b]) — *Soit $X \in \mathbb{R}^{n\times p}$ une matrice de rang plein. On suppose que X satisfait* $\mathrm{UDP}(S_0, \kappa_0, \Delta)$. *Soit $\lambda_\ell \in \mathbb{R}$ tel que*

$$\lambda_\ell > \lambda^0/(1-2\kappa_0), \tag{II.9}$$

où λ^0 est un paramètre qui ne dépend que de X et du bruit z, voir page 21. On a

$$\|\beta^\ell - \beta^\star\|_{\ell_1} \leq \frac{2}{(1-\frac{\lambda^0}{\lambda_\ell}) - 2\kappa_0} \min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\ |\mathcal{S}|=s,\ s\leq S_0.}} \Big(\lambda_\ell\,\Delta^2\, s + \|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big)\,,$$

$$\|X\beta^\ell - X\beta^\star\|_{\ell_2} \leq \min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\ |\mathcal{S}|=s,\ s\leq S_0.}} \Big[4\lambda_\ell\,\Delta\sqrt{s} + \frac{\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}}{\Delta\sqrt{s}}\Big]\,,$$

On a le résultat plus faible en terme de la distorsion :

$$\|\beta^\ell - \beta^\star\|_{\ell_1} \leq \frac{2}{(1-\frac{\lambda^0}{\lambda_\ell}) - 2\kappa_0} \min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\ |\mathcal{S}|=s,\\ s\leq(\kappa_0/\delta)^2 p.}} \Big(\lambda_\ell\cdot\frac{4\,\delta^2}{\rho_n^2}\cdot s + \|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big)\,,$$

$$\|X\beta^\ell - X\beta^\star\|_{\ell_2} \leq \min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\ |\mathcal{S}|=s,\\ s\leq(\kappa_0/\delta)^2 p.}} \Big[4\lambda_\ell\cdot\frac{2\,\delta}{\rho_n}\cdot\sqrt{s} + \frac{1}{2\delta\sqrt{s}}\cdot\rho_n\,\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big]\,,$$

où ρ_n est la plus petite valeur singulière de X.

Le même type de résultat est vrai pour le sélecteur Dantzig, voir les théorèmes II.6 et II.7 à la page 27. Ces théorèmes montrent que toute construction déterministe de sous-espaces presque-euclidiens donne des designs dont les inégalités oracles sont décrites ci-dessus.

2. Designs construits à partir de graphes expanseurs

La condition UDP n'est pas liée qu'aux seuls sous-espaces presque-Euclidiens, on montre qu'elle est satisfaite par toutes les matrices d'adjacence renormalisées de graphes expanseurs.

Définition 2.1 (Graphe (s,ε)-expanseur) — *Un graphe (s,ε)-expanseur est un graphe simple biparti $G = (A, B, E)$ de degré à gauche d et tel que pour tout $\Omega \subset A$ avec $|\Omega| \leq s$, l'ensemble des voisins $\Gamma(\Omega)$ de Ω ait une taille satisfaisant*

$$|\Gamma(\Omega)| \geq (1-\varepsilon)\, d\, |\Omega| \;. \tag{III.2}$$

Le paramètre ε est appelée la constante d'expansion.

La matrice d'adjacence renormalisée d'un graphe simple biparti G de degré à gauche d est définie par :

$$X_{ij} = \begin{cases} 1/d & \text{si } i \text{ est connecté à } j\,, \\ 0 & \text{sinon}\,, \end{cases} \tag{III.1}$$

où $i \in \{1,\dots,n\}$ et $j \in \{1,\dots,p\}$. On montre alors le lemme suivant.

Lemme 2.1 — *Soit $X \in \mathbb{R}^{n\times p}$ la matrice d'adjacence renormalisée d'un graphe $(2s,\varepsilon)$-expanseur tel que $\varepsilon < 1/10$. Alors X satisfait $\mathrm{UDP}(s,\kappa_0,\Delta)$ avec*

$$\kappa_0 = \frac{2\varepsilon}{1-2\varepsilon} \quad \text{and} \quad \Delta = \frac{\sqrt{n}}{(1-2\varepsilon)\sqrt{s}}\,.$$

On remarque que κ_0 est strictement plus petit que $1/4$ pour $\varepsilon < 1/10$.

On en déduit des inégalités oracles pour les matrices d'adjacence des graphes expanseurs.

Théorème 2.2 ([dC10]) — *Soit $X \in \mathbb{R}^{n\times p}$ une matrice d'adjacence renormalisée d'un graphe $(2s,\varepsilon)$-expanseur avec $\varepsilon \leq 1/12$. Alors, pour tout*

$$\lambda_\ell > 5\lambda^0/3\,, \tag{III.8}$$

où λ^0 est un paramètre qui ne dépend que de X et du bruit z, on a

$$\|\beta^\ell - \beta^\star\|_{\ell_1} \leq \frac{2}{\left(1-\frac{\lambda^0}{\lambda_\ell}\right)-\frac{2}{5}} \cdot \left[\frac{36}{25}\cdot\lambda_\ell\cdot n + \min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\ |\mathcal{S}|=s.}} \|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\right]. \tag{III.9}$$

avec grande probabilité. De même, on a

$$\|X\beta^\ell - X\beta^\star\|_{\ell_2} \leq \frac{24}{5}\cdot\lambda_\ell\cdot\sqrt{n} + \frac{5}{6\sqrt{n}}\cdot \min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\ |\mathcal{S}|=s.}} \|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}. \tag{III.10}$$

avec grande probabilité.

Le même type de résultats sont montrés pour le sélecteur Dantzig, voir à la page 41.

3. Designs issus de la reconstruction de mesures signées

Dans le chapitre IV, ce mémoire montre que l'on peut étendre, très naturellement, les idées du Compressed Sensing à un espace de dimension infinie. Certes, de très beaux travaux ont été menés sur ce sujet. Citons par exemple, les travaux de A.C. Hansen [Han11] sur le « Compressed Sensing infini » qui s'intéressent à des espaces hilbertiens. Cependant, notre approche est différente puisqu'on s'intéresse à l'espace de Banach des mesures signées sur la droite réelle. Cette approche nouvelle mène à la construction de designs déterministes optimaux très simples. Plus précisément, il est aisé de voir que le nombre minimal de mesures nécessaires pour reconstruire fidèlement tout signal s-parcimonieux (à partir d'une observation non-bruitée $y = X\beta^\star$) est $n_{opt} = 2s$. En effet, si la matrice de design X a moins de $2s$ lignes alors il est possible de trouver deux vecteurs s-parcimonieux ayant la même image. Ainsi, il sera impossible de reconstruire ces vecteurs quelque soit la méthode utilisée. Dans le chapitre IV, on montre qu'il existe toute une famille de designs pour lesquels le basis pursuit reconstruit exactement tout vecteur s-parcimonieux à partir de seulement $n = 2s + 1$ variables d'observation. Ce résultat n'est pas nouveau et est bien connu de la communauté des polytopes convexes. En particulier, D.L. Donoho et J. Tanner [DT05] ont montré un résultat similaire (bien que plus restrictif) en étudiant la dualité entre la reconstruction des polytopes convexes k-voisinants et le Compressed Sensing. On souligne que notre approche est étrangère à la leur et nos résultats s'appliquent à une famille de designs plus générale.

Notre point de vue est la reconstruction fidèle des mesures signées de support fini. Soit I un borélien de $\mathbb{R}$ que l'on suppose borné. Soit σ une mesure de support fini inclus dans I :

$$\sigma = \sum_{i=1}^{s} \sigma_i \delta_{x_i} \,,$$

où σ_i sont les poids et δ_x est la masse de Dirac en x. On souligne que ni les poids, ni les points x_i ne sont connus. On observe $n+1$ moments généralisés :

$$c_k(\sigma) = \int_I u_k \,\mathrm{d}\sigma \,, \tag{IV.1}$$

où $k = 0, \ldots, n$ et $\mathcal{F} := \{u_0, \ldots, u_n\}$ est une famille de fonctions continues sur la fermeture de I. On suppose que $\mathcal{F}$ est un M-système homogène. La définition précise se trouve à la page 52. On rappelle que ces derniers englobent les moments standards, la transformation de Stieltjes, la transformée de Laplace, la fonction caractéristique... Dans ce mémoire, on introduit le support pursuit [dCG11]. Il est défini par

$$\sigma^\star \in \operatorname*{Arg\,min}_{\mu \in \mathcal{M}} \|\mu\|_{TV} \quad \text{tel que } \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma) \,, \tag{IV.2}$$

où $\|\,.\,\|_{TV}$ est la norme de variation totale, $\mathcal{K}_n(\mu) := \{c_0(\mu), \ldots, c_n(\mu)\}$, et $\mathcal{M}$ représente l'ensemble des mesures signées sur I. Contrairement à de précédents travaux [KN77, GG96], on souligne que cet estimateur considère toutes les mesures *signées* aux $n+1$ premiers moments fixés.

Théorème 3.1 ([dCG11]) — *Soit $\mathcal{F}$ un M-système homogène sur I. Soit σ un mesure positive de support fini inclus dans I. Alors σ est l'unique solution du support pursuit étant donné l'observation $\mathcal{K}_n(\sigma)$ dès lors que n est strictement plus grand deux fois la taille du support de σ.*

Ce résultat donne toute une famille de designs déterministes pour le Compressed Sensing. On rappelle que le basis pursuit est défini par

$$\beta^{bp} \in \arg\min_{X\beta=y} \|\beta\|_{\ell_1} . \tag{I.11}$$

Une étude complète de cet estimateur est menée au chapitre I. L'étude des mesures signées montre alors le résultat suivant.

Théorème 3.2 ([**DT05**, **dCG11**]) — *Soient n, p, s des entiers tels que*

$$s \leq \min(n/2,\, p).$$

Soit $\{1, u_1, \ldots, u_n\}$ un M-système homogène sur I. Soient $t_1, \ldots, t_p$ des réels distincts deux à deux de I. Soit X un système de Vandermonde généralisé

$$X = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ u_1(t_1) & u_1(t_2) & \ldots & u_1(t_p) \\ u_2(t_1) & u_2(t_2) & \ldots & u_2(t_p) \\ \vdots & \vdots & & \vdots \\ u_n(t_1) & u_n(t_2) & \ldots & u_n(t_p) \end{pmatrix} .$$

Alors basis pursuit (I.11) *reconstruit fidèlement tout vecteur s-parcimonieux $\beta^\star \in \mathbb{R}^p$ à partir de l'observation $X\beta^\star$.*

Dans un second temps, le chapitre IV s'intéresse à la reconstruction fidèle de mesures cibles dont les poids sont de signes quelconques. Notre approche est basée sur une condition suffisante qui généralise la notion de *certificat dual* [**CP10**] du Compressed Sensing.

En résumé, les quatre premiers chapitres de cette thèse donnent et étudient trois façons de construire des designs déterministes en grandes dimensions :

- ✧ en utilisant une section presque-Euclidienne de la boule unité ℓ_1 (chapitre II),
- ✧ en utilisant des graphes expanseurs (chapitre III),
- ✧ en utilisant une matrice de Vandermonde généralisée (chapitre IV).

À notre connaissance, les deux premières constructions n'ont pas été étudiées en détail dans le cadre du lasso et du sélecteur Dantzig. Elles le sont dans deux articles de l'auteur, à savoir [**dC11b**] et [**dC10**].

4. Autres travaux

Dans le dernier chapitre de ce mémoire, on s'intéresse à des inégalités isopérimétriques quantitatives optimales sur la droite réelle et l'on prouve que parmi les ensembles de mesure donnée et d'asymétrie donnée (distance à la demi-droite, i.e. distance aux ensembles de périmètre minimal), les intervalles ou les complémentaires d'intervalles ont le plus petit périmètre. Les travaux exposés utilisent uniquement des outils géométriques et complètent le remarquable résultat [**CFMP11**] en précisant la stabilité des inégalités isopérimétriques dans le cas des mesures log-concaves symétriques sur la droite réelle. Ces travaux sont étudiés dans [**dC11a**].

Table des matières

Table des figures

CHAPITRE I

Régression linéraire en grandes dimensions

Notre volonté de mieux comprendre des mécanismes complexes dépendants d'un grand nombre de variables explicatives est une formidable source de motivation pour la communauté statistique. On peut par exemple penser à des recherches de premier plan comme

- ✧ la génétique : trouver les locations du génome qui influencent un trait particulier (e.g. le taux de cholestérol),
- ✧ l'imagerie médicale : reconstruction parfaite d'images IRM sous-échantillonnées [CRT06a, DLP07],
- ✧ le biosensing avec les travaux fondateurs de T.R. Golub [BCC+99] et l'émergence de l'analyse micro-array,
- ✧ l'analyse des données géophysiques [HHS07],
- ✧ l'astronomie [BOS08], ...

Les applications sont aussi variées que le nombre de paramètres qu'elles font intervenir. Cependant elles ont un point commun : le modèle linéaire. Ce chapitre introductif en présente les problématiques et les outils fondamentaux, de la régression ridge au lasso [Tib96] en 1996 et à l'émergence du Compressed Sensing [Don06] en 2004.

1. Le modèle linéaire

Nous parlons ici du modèle linéaire en toute généralité en gardant à l'esprit que ce modèle peut (doit) être reproduit dans de nombreuses applications. Le vocabulaire standard est emprunté au monde expérimental, on parlera d'expérimentateur, de plans d'expériences, ... et il sera vital de faire la différence entre les quantités connues de l'expérimentateur et celles que l'on cherche à estimer/prédire. Le modèle linéaire est défini comme suit :

$$y = X\beta^\star + z\,, \tag{I.1}$$

où $y \in \mathbb{R}^n$ est l'observation (ou réponse), $X \in \mathbb{R}^{n\times p}$ est la matrice de design, $\beta^\star \in \mathbb{R}^p$ est la cible, et $z \in \mathbb{R}^n$ est un terme d'erreur appelé bruit. Précisons le vocabulaire ici.

- ✦ **Observation:** Le vecteur $y \in \mathbb{R}^n$ est le résultat observé d'une expérience statistique. Ses coefficients y_i sont appelés les variables endogènes, variables réponses, variables mesurées, ou encore variables dépendantes.
- ✦ **Design:** Il est connu (et parfois choisi) par l'expérimentateur pour déterminer la cible.
- ✦ **Cible:** Elle est au cœur de l'estimation statistique. Elle représente les variables d'intérêt. Les entrées de $\beta^\star \in \mathbb{R}^p$ sont appelées les coefficients de régression.
- ✦ **Bruit:** Le vecteur $z \in \mathbb{R}^n$ capture les autres facteurs qui perturbent l'observation. Selon le modèle, z peut être un vecteur aléatoire distribué selon une loi connue. Dans le cas où le bruit est distribué selon une loi normale centrée, on

parle de modèle linéaire Gaussien :

$$z \sim \mathcal{N}_n\big(0, \sigma_n^2 \,\mathrm{Id}_n\big)\,, \tag{I.2}$$

où $\mathcal{N}_n$ est la distribution Gaussienne n-multivariée, $\mathrm{Id}_n \in \mathbb{R}^{n\times n}$ est la matrice identité, et $\sigma_n > 0$ est l'écart type.

1.1. Les moindres carrés. Dans le cas où le nombre d'observations n est plus grand que le nombre de prédicteurs p, l'estimateur classique est alors l'estimateur des moindres carrés :

$$\beta^{ls} \in \arg\min_{\beta\in\mathbb{R}^p} \|y - X\beta\|_{\ell_2}^2, \tag{I.3}$$

où $\|\,.\,\|_{\ell_2}$ est la norme euclidienne de $\mathbb{R}^n$ définie par

$$\forall x \in \mathbb{R}^n, \quad \|x\|_{\ell_2}^2 = \sum_{i=1}^n x_i^2.$$

On sait alors que les moindres carrés ont une unique solution si et seulement si la matrice de design X a un noyau réduit au vecteur nul. Dans ce cas, on peut donner la forme explicite de l'estimateur des moindres carrés.

Proposition I.1 — *Supposons que la matrice de design ait un noyau réduit au vecteur nul, alors l'unique minimiseur des moindres carrés (I.3) est donné par*

$$\beta^{\ell s} = \big(X^\top X\big)^{-1} X^\top y. \tag{I.4}$$

où $X^\top$ est la matrice transposée de la matrice X. Dans le cas du modèle linéaire Gaussien, on a

$$\beta^{\ell s} \sim \mathcal{N}_p\big(\beta^\star, \sigma_n^2 \big(X^\top X\big)^{-1}\big)\,. \tag{I.5}$$

Démonstration. Remarquons que $\ker(X^\top X) = \ker(X)$, on vérifie alors que $X^\top X$ est inversible si et seulement si X est injective. Au point $\beta^{\ell s}$, le gradient de la fonction objectif $\beta \mapsto \|y - X\beta\|_{\ell_2}$ est nul, ce qui s'écrit encore

$$X^\top X\beta^{\ell s} - X^\top y = 0.$$

Cette expression donne (I.4). Dans le cas Gaussien, on a $y \sim \mathcal{N}_n(X\beta^\star, \sigma_n^2 \,\mathrm{Id}_n)$, l'expression (I.5) suit. □

La matrice $X^\top X$ est appelée *la matrice d'information*, elle représente la matrice de covariance de l'estimateur des moindres carrés dans le cas Gaussien. Dans le cas où la matrice de design n'est pas injective (on dira alors qu'elle est singulière), il y a une infinité de solutions aux moindres carrés.

Proposition I.2 — *Si X est singulière alors l'ensemble $\mathfrak{A}^{ls}$ de tous les minimiseurs des moindres carrés (I.3) est l'espace affine*

$$\mathfrak{A}^{\ell s} = \beta^{\ell s} + \ker(X)\,,$$

où $\beta^{\ell s} \in \mathbb{R}^p$ est n'importe quelle solution de (I.3).

Démonstration. Soit $\beta^{\ell s} \in \mathbb{R}^p$ une solution de (I.3). Observons que $X\beta^{\ell s}$ est l'unique projection Euclidienne de y sur le sous-espace vectoriel engendré par les vecteurs X_i (i.e. les colonnes de X). Le résultat suit. □

1.2. La régression ridge. On s'intéresse au modèle sous-déterminé où n est strictement inférieur à p. Dans ce cas, la matrice de design est singulière et on ne peut raisonnablement pas utiliser les moindres carrés. Une solution est alors de les « régulariser » en ajoutant une pénalité rendant la fonction objectif strictement convexe. L'estimateur ridge (Hoerl et Kennard, 1970) suit cette démarche, il est défini par :

$$\beta^r \in \arg\min_{\beta\in\mathbb{R}^p} \{\|y - X\beta\|_{\ell_2}^2 + \lambda_r \|\beta\|_{\ell_2}^2\}, \tag{I.6}$$

où $\lambda_r > 0$ est la *constante ridge*. Le problème (I.6) a une unique solution. En effet, le gradient de la fonction objectif $\beta \mapsto \|y - X\beta\|_{\ell_2}^2 + \lambda_r \|\beta\|_{\ell_2}^2$ est nul uniquement au point β^r tel que

$$\big(X^\top X + \lambda_r \operatorname{Id}_p\big)\beta^r - X^\top y = 0.$$

Observons que même si $X^\top X$ est singulière, $X^\top X + \lambda \operatorname{Id}_p$ est inversible. On a donc le résultat suivant.

Proposition I.3 — *Le programme* (I.6) *a une unique solution, à savoir*

$$\beta^r = \big(X^\top X + \lambda_r \operatorname{Id}_p\big)^{-1} X^\top y\,, \tag{I.7}$$

De plus dans le cas Gaussien, on a

$$\beta^r \sim \mathcal{N}_p\left(S_{\lambda_r} X^\top X \beta^\star,\, \sigma_n^2 S_{\lambda_r}(X^\top X) S_{\lambda_r}\right)\,,$$

où $S_{\lambda_r} = \big(X^\top X + \lambda_r \operatorname{Id}_p\big)^{-1}$.

Ainsi, pour n'importe quelles valeurs de n, p et λ_r, et n'importe quel type de bruit z, la régression ridge (I.6) a une unique solution. D'un point de vue de la théorie de l'optimisation, il est bien connu que le programme (I.6) peut être résolu à l'aide de deux autres programmes équivalents, voir la figure 1.1.

Remarque — Considérons une observation $y \in \mathbb{R}^n$. Soit $\lambda_r > 0$ et posons

- ✧ $\beta^r = \big(X^\top X + \lambda_r \operatorname{Id}_p\big)^{-1} X^\top y\,,$
- ✧ $\mu_r = \|\beta^r\|_{\ell_2}^2\,,$
- ✧ $\nu_r = \|y - X\beta^r\|_{\ell_2}^2\,.$

Alors les trois programmes suivants ont la même (unique) solution β_r, à savoir

- ✧ $\beta^r = \arg\min_{\beta\in\mathbb{R}^p} \{\|y - X\beta\|_{\ell_2}^2 + \lambda_r \|\beta\|_{\ell_2}^2\}\,,$
- ✧ $\beta^r = \arg\min_{\|\beta\|_{\ell_2}^2 \le \mu_r} \|y - X\beta\|_{\ell_2}^2\,,$
- ✧ $\beta^r = \arg\min_{\|y - X\beta\|_{\ell_2}^2 \le \nu_r} \|\beta\|_{\ell_2}^2\,.$

Les relations entre les paramètres λ_r, μ_r, et ν_r dépendent de l'observation y, plus précisément on a

$$\mu_r = \big\|\big(X^\top X + \lambda_r \operatorname{Id}_p\big)^{-1} X^\top y\big\|_{\ell_2}^2\,, \tag{I.8a}$$

$$\nu_r = \big\|\big(\operatorname{Id}_n - X\big(X^\top X + \lambda_r \operatorname{Id}_p\big)^{-1} X^\top\big)y\big\|_{\ell_2}^2\,. \tag{I.8b}$$

Comme y est un vecteur aléatoire (à $\beta^\star$ fixé), il n'y a aucun espoir de pouvoir exprimer les paramètres μ_r et ν_r en fonction seulement de λ_r et $\beta^\star$. En effet, ils sont des variables aléatoires (cf (I.8)). En pratique, il n'y a donc aucune relation exploitable entre ces paramètres.

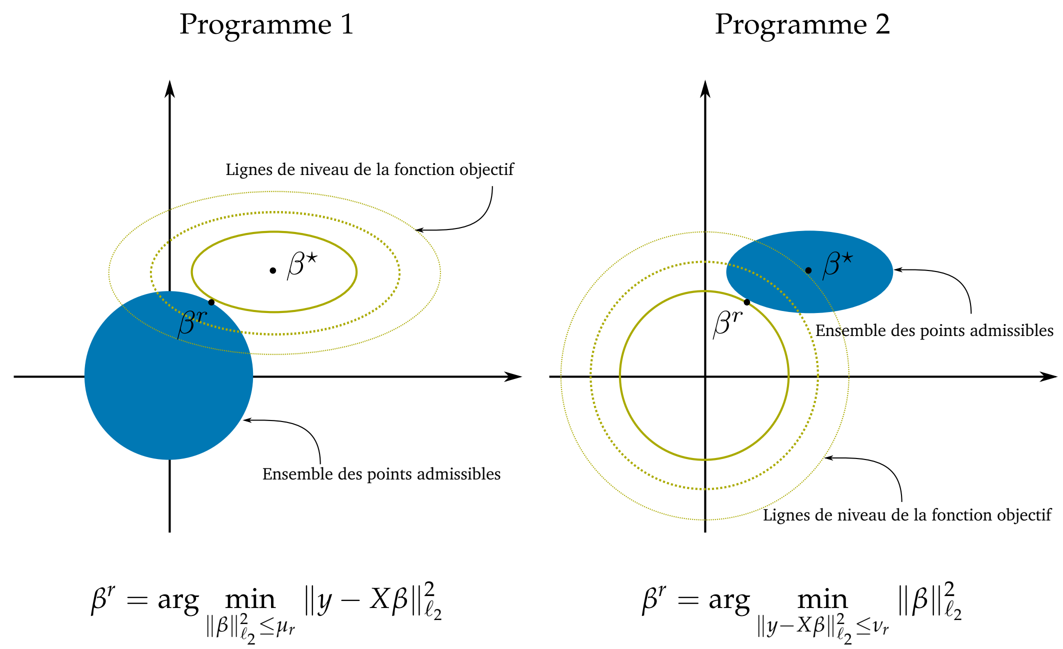


FIGURE 1.1. Les cercles représentent les lignes de niveau de la norme Euclidienne, tandis que les ellipses représentent celles du critère quadratique $\|y - X\beta\|_{\ell_2}^2$. Dans chaque programme, l'ensemble des points admissibles est représenté en bleu. L'estimateur ridge β^r est au point de contact entre une surface lisse (donnée par une somme de carrés) et d'une boule Euclidienne. Quand $\mu_r = \|\beta^r\|_{\ell_2}^2$ et $\nu_r = \|y - X\beta^r\|_{\ell_2}^2$, les deux programmes ont la même solution.

Observons enfin que la régression ridge est une méthode de « shrinkage ». En effet, considérons la décomposition en valeurs singulières (SVD) de X, à savoir

$$X = UDV^\top ,$$

où $U \in \mathbb{R}^{n\times n}$ et $V \in \mathbb{R}^{p\times p}$ sont des matrices orthogonales, et $D \in \mathbb{R}^{n\times p}$ est une matrice diagonale aux entrées $d_1 \geq \cdots \geq d_n \geq 0$. En utilisant la SVD, la prédiction ridge s'écrit

$$\begin{aligned} X\beta^r &= X\big(X^\top X + \lambda_r \mathrm{Id}_p\big)^{-1} X^\top y\,, \\ &= UD\big(D^\top D + \lambda_r \mathrm{Id}_p\big)^{-1} D^\top U^\top y\,, \\ &= \sum_{i=1}^n u_i \,\frac{d_i^2}{d_i^2 + \lambda_r}\, u_i^\top y\,, \end{aligned}$$

où les u_i sont les colonnes de U. Comme $\lambda_r > 0$, il vient que

$$\frac{d_i^2}{d_i^2 + \lambda_r} < 1.$$

La régression ridge peut être comprise comme un algorithme en deux étapes. D'abord on calcule les coordonnées de y dans la base U. Puis on les réduit (« shrink » en anglais) toutes d'un facteur $d_i^2/(d_i^2 + \lambda_r)$. Ainsi les coordonnées d_i ayant de petites valeurs seront les plus « écrasées ». Pour résumer, la régression ridge favorise les grandes composantes principales (au sens de la valeur absolue des valeurs propres) au détriment des petites. Cette méthode présente plusieurs inconvénients. Tout d'abord, on vient de voir que cet estimateur favorise des directions de l'espace déterminées par le design et non pas par la cible. De plus, cet estimateur ne préserve aucune structure forte telle que la parcimonie (i.e. la cible à un « petit » support). Pour palier à cela, on s'intéressera à des fonctions objectif non différentiables, présentant une singularité sur l'espace des vecteurs « parcimonieux ».

2. La régression parcimonieuse

Dans le cas où $n \ll p$, le système $y = X\beta^\star$ est fortement sous-déterminé. Sans plus de structure sur le vecteur $\beta^\star$, il est impossible de le reconstruire exactement à partir de la seule observation y. Pour contourner ce problème, nous nous intéressons aux vecteurs parcimonieux.

Définition I.1 (Le modèle parcimonieux) — *Dans le modèle parcimonieux, le vecteur cible $\beta^\star$ est s-parcimonieux, i.e. la taille de son support ne dépasse pas s. Le paramètre s est appelé constante de parcimonie.*

Le modèle parcimonieux est au cœur de nombreuses recherches depuis un peu moins d'une dizaine d'années. Il est apparu qu'un grand nombre d'applications font intervenir le modèle parcimonieux. Le plus pédagogique est peut-être celui de l'acquisition et la compression simultanée d'images. Il est bien connu qu'une image tirée d'un capteur ayant 15 millions de pixels peut être compressée (avec une perte tout à fait raisonnable) en une centaine de milliers de coefficients d'ondelettes. D'aucun n'a qu'à penser au format JPEG pour se convaincre de cela. Ainsi les images courantes peuvent être vues comme des vecteurs parcimonieux dans une base d'ondelettes. Bien évidement cela n'est pas le seul exemple [**Don06**], le modèle parcimonieux est particulièrement pertinent en génomique, en imagerie par résonance magnétique, en recherche de champs de pétrole par réflexion sismique, ... Pour des raisons pédagogiques et afin d'introduire les outils du modèle parcimonieux, nous commençons par étudier une version sans bruit ($z = 0$) du modèle linéaire (I.1), il s'écrit alors :

$$y = X\beta^\star\,. \tag{I.9}$$

Nous supposons désormais que la cible $\beta^\star$ est s-parcimonieuse. Le vecteur admissible (i.e.. satisfaisant la contrainte (I.9)) le plus parcimonieux est donné par l'*estimateur combinatoire* :

$$\beta^c \in \arg\min_{X\beta=y} \|\beta\|_{\ell_0}\,, \tag{I.10}$$

où $\|\,.\,\|_{\ell_0}$ est la « norme » ℓ_0 standard,

$$\forall x \in \mathbb{R}^p, \quad \|x\|_{\ell_0} = \sharp\{x_i \mid x_i \neq 0\}.$$

Cet estimateur naïf donne la solution la plus parcimonieuse. Dans la pratique, il est impossible de calculer cet estimateur pour des valeurs de p plus grandes que quelques dizaines. En effet, la seule méthode connue pour résoudre ce programme est de faire varier $\mathcal{S}$ parmi tous les sous-ensembles $\mathcal{S} \subset \{1,\ldots,p\}$ possibles et de vérifier si y est une combinaison linéaire des X_i pour $i \in \mathcal{S}$. Ce programme est NP-difficile est ne peut être résolu en pratique. Une alternative est fournie par relaxation convexe. En 1998, S.S. Chen, D.L. Donoho, et M. A. Saunders [**CDS98**] ont introduit le *basis pursuit* comme suit

$$\beta^{bp} \in \arg\min_{X\beta=y} \|\beta\|_{\ell_1}\,, \tag{I.11}$$

où $\|.\|_{\ell_1}$ est la norme ℓ_1 standard :

$$\forall x \in \mathbb{R}^m, \quad \|x\|_{\ell_1} = \sum_{i=1}^{m} |x_i|\,.$$

L'avantage de ce programme est qu'il est équivalent au programme linéaire suivant :

$$(\gamma^\star, t^\star) \in \arg \min_{\substack{X\gamma=y \\ -t_i \leq \gamma_i \leq t_i}} \sum_{i=1}^{p} t_i\,,$$

et l'on peut vérifier que la solution satisfait $\gamma^\star = \beta^{bp}$ et $t_i^\star = |\beta_i^{bp}|$ pour tout $i = 1, \ldots, p$. Ce programme peut être résolu très efficacement en utilisant soit l'algorithme du simplexe [Mur83], soit la méthode du point intérieur [Kar84]. On en parlera pas de la résolution numérique de cet algorithme dans cette thèse. Le lecteur intéressé par cette problématique trouvera tout le matériel nécessaire dans [Dan98, BV04].

2.1. Le certificat dual. Le certificat dual est une condition suffisante pour garantir qu'un vecteur parcimonieux donné soit l'unique solution du basis pursuit. On commence par quelques notations.

- ✧ Soit $\mathcal{S} \subset \{1, \ldots, p\}$ un sous-ensemble de taille s, on note $X_\mathcal{S} \in \mathbb{R}^{n\times s}$ la sous-matrice de X dont les colonnes X_i sont telles que $i \in \mathcal{S}$.
- ✧ On dit qu'un vecteur $\eta \in \mathbb{R}^p$ est dans l'*image duale* de X s'il existe $\lambda \in \mathbb{R}^n$ tel que $\eta = X^\top \lambda$.
- ✧ Pour tout vecteur $\gamma \in \mathbb{R}^p$ et tout sous-ensemble $E \subseteq \{1, \ldots, p\}$, on note $\gamma_E \in \mathbb{R}^p$ le vecteur dont la i-ème coordonnée vaut γ_i si $i \in E$ et 0 sinon.
- ✧ Pour tout vecteur $\gamma \in \mathbb{R}^p$, on note $\mathrm{sgn}(\gamma) \in \mathbb{R}^p$ le vecteur dont la i-ème coordonnée vaut $+1$ si $\gamma_i \geq 0$ et -1 sinon.

La proposition suivante donne une condition suffisante pour que le vecteur cible $\beta^\star$ soit l'unique solution du basis pursuit.

Proposition I.4 ([CRT06a, CP10], Certificat dual exact) — *Soit $\beta^\star \in \mathbb{R}^p$ un vecteur s-parcimonieux et $\mathcal{S}$ son support. Supposons que $X_\mathcal{S}$ soit de rang s. Si on peut trouver $\eta \in \mathbb{R}^p$ satisfaisant le problème d'interpolation suivant :*

(i) *η est dans l'image duale de X,*
(ii) $\eta_\mathcal{S} = \mathrm{sgn}(\beta^\star)_\mathcal{S}$,
(iii) $\|\eta_{\mathcal{S}^c}\|_\infty < 1$.

Alors $\beta^\star$ est l'unique solution du basis pursuit (I.11) étant donné l'observation $y = X\beta^\star$.

Démonstration. Soit $\beta \in \mathbb{R}^p$ un point admissible (i.e. $X\beta = X\beta^\star$) différent de $\beta^\star$. On note $h = \beta^\star - \beta$. Ce dernier appartient au noyau de X. Comme $\eta \in \mathbb{R}^p$ est dans l'image duale de X, il existe $\lambda \in \mathbb{R}^n$ tel que $\eta = X^\top\lambda$. Cela donne

$$\begin{aligned} 0 &= \lambda^\top X h = h^\top X^\top \lambda = h^\top \eta \\ &= \sum_{i\in\mathcal{S}} \mathrm{sgn}(\beta_i^\star)(\beta_i^\star - \beta_i) - \sum_{i\in\mathcal{S}^c} \eta_i\beta_i \\ &\geq \|\beta^\star\|_{\ell_1} - \|\beta_\mathcal{S}\|_{\ell_1} - \sum_{i\in\mathcal{S}^c} \eta_i\beta_i \end{aligned} \tag{I.12}$$

On distingue deux cas :

- ✧ Il existe $j \in \mathcal{S}^c$ tel que $\beta_j \neq 0$ alors l'inégalité de Hölder donne $\sum_{i\in\mathcal{S}^c} \eta_i\beta_i < \|\beta_{\mathcal{S}^c}\|$.

  L'inégalité (I.12) donne alors $\|\beta^\star\|_{\ell_1} < \|\beta\|_{\ell_1}$.

✧ Sinon, le support de β est inclus dans $\mathcal{S}$. Comme $X_{\mathcal{S}}$ est de rang s, la contrainte $X\beta = X\beta^\star$ donne $\beta = \beta^\star$.

Cela termine la preuve. □

La preuve donne un résultat plus fort : Si un certificat dual η existe alors le basis pursuit reconstruit exactement tout vecteur β tel que :

(1) son support T est inclus dans l'ensemble $\mathcal{S}$ des indices i pour lesquels $\eta_i = \pm 1$,

(2) et $\mathrm{sgn}(\beta)_T = \eta_T$.

Le sous-ensemble de la sphère unité ℓ_1 composé des vecteurs satisfaisant ces deux conditions forme une $(s-1)$-facette, où s est le cardinal de $\mathcal{S}$. On remarque que le certificat dual η est un sous-gradient de la norme ℓ_1 en tout point de la $(s-1)$-facette. De même, les coefficients λ_i tels que $\eta = X^\top \lambda$ peuvent être vus comme des multiplicateurs KKT [KT51]. Ils montrent en particulier que le vecteur η est perpendiculaire au noyau de X. Pour résumé, la proposition I.4 garantit que le plan affine $\{\gamma \mid X\gamma = X\beta^\star\}$ soit tangent à la sphère unité ℓ_1 au point $\beta^\star$, voir la figure 1.2.

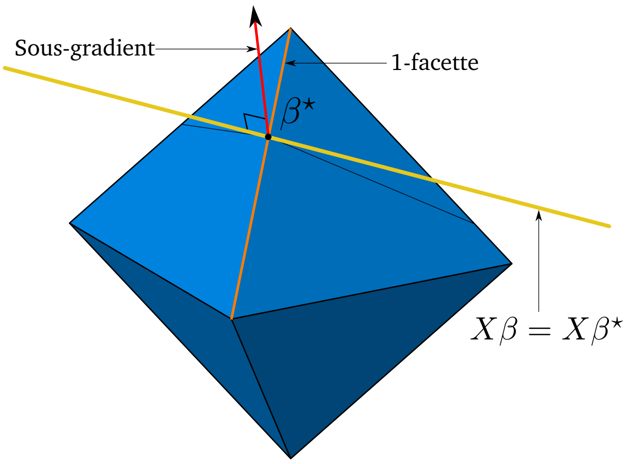


FIGURE 1.2. Le certificat dual est un sous-gradient de la norme ℓ_1 perpendiculaire à l'espace affine $\{\beta \mid X\beta = X\beta^\star\}$. En orange, on a représenté la 1-facette à laquelle appartient le vecteur cible. En rouge est représenté le sous-gradient (certificat dual). L'espace affine jaune correspond à l'ensemble des points admissibles. Dans cette configuration, le certificat dual non seulement garantit la reconstruction exacte de $\beta^\star$ mais aussi de tous les vecteurs de la 1-facette.

Le certificat dual est un outil précieux pour étudier le basis pursuit. Il s'étend naturellement à des espaces de Banach comme par exemple l'espace des mesures signées sur la droite réelle (voir Chapitre IV).

2.2. La Nullspace Property. Cette propriété est une condition nécessaire et suffisante sur la reconstruction exacte des vecteurs parcimonieux. Elle a été introduite par A. Cohen, W. Dahmen et R. DeVore [CDD09].

Définition I.2 (NSP, Nullspace Property) — *Soit $X \in \mathbb{R}^{n\times p}$ une matrice. On dit que X satisfait la propriété* NSP *d'ordre s si et seulement si pour tout vecteur non-nul h dans le noyau de X, et pour tout sous-ensemble d'indices S de taille s, on a*

$$\|h_S\|_{\ell_1} < \|h_{S^c}\|_{\ell_1} .$$

Cette propriété signifie que plus de la moitié du poids de h, à savoir $\|h\|_{\ell_1}/2$, ne peut être concentré sur un « petit » sous-ensemble. Cette propriété peut être vue comme un principe d'incertitude sur le noyau de la matrice de design.

Proposition I.5 — *Soit s un entier tel que $0 < s \leq p$. Les deux assertions suivantes sont équivalentes :*

- (i) *La matrice de design X satisfait la propriété* NSP *d'ordre s ;*
- (ii) *Pour tout vecteur s-parcimonieux $\beta^\star$, le basis pursuit (I.11) a pour unique solution $\beta^\star$ étant donné l'observation $y = X\beta^\star$.*

Démonstration. Supposons que (i) est vrai. Soit $\beta^\star$ un vecteur s-parcimonieux et notons $\mathcal{S}$ son support. Soit $\beta \in \mathbb{R}^p$ un point admissible (i.e. $X\beta = y$). Supposons que $\beta^\star \neq \beta$. En notant $h = \beta - \beta^\star$ on a alors que h est un vecteur non-nul du noyau de X. En utilisant la propriété NSP, il vient que

$$\|\beta\|_{\ell_1} = \|\beta^\star + h_S\|_{\ell_1} + \|h_{S^c}\|_{\ell_1} \geq \|\beta^\star\| - \|h_S\|_{\ell_1} + \|h_{S^c}\|_{\ell_1} > \|\beta^\star\|_{\ell_1},$$

où S est le support de $\beta^\star$. Cela donne (ii).

Réciproquement, on suppose que (ii) est vrai. Soit $\gamma \in \mathbb{R}^p$ un vecteur non-nul du noyau de X. Soit $S \subset \{1, \ldots, p\}$ tel que $|S| \leq s$. On pose $\beta^\star = -\gamma_S$. Comme $\beta^\star$ est l'unique solution du basis pursuit et que $\gamma_{S^c} = \beta^\star + \gamma$ est un point admissible, il s'en suit

$$\|\gamma_{S^c}\|_{\ell_1} = \|\beta^\star + \gamma\|_{\ell_1} > \|\beta^\star\|_{\ell_1} = \|\gamma_S\|\,.$$

Ce qui conclut la preuve. □

Cette propriété est surprenante, elle montre que les performances du basis pursuit ne dépendent que du noyau de la matrice de design. On peut préciser cette pensée en étudiant la largeur de Gelfand de la boule unité ℓ_1.

2.3. La largeur de Gelfand. La largeur de Gelfand est une notion de la théorie de l'approximation dans les espaces de Banach. Elle permet de comparer deux normes sur tous les sous-espaces vectoriels de tailles majorées par une constante. Plus précisément, la largeur de Gelfand n-dimensionnelle de la boule unité ℓ_1 de $\mathbb{R}^p$ par rapport à la norme ℓ_2 est définie par :

$$d^n(B_1^p, \ell_2) = \inf_{\substack{\Gamma_n \subseteq \mathbb{R}^p \\ \mathrm{codim}(\Gamma_n) \leq n}} \ \sup_{x \in \Gamma_n \cap B_1^p} \|x\|_{\ell_2},$$

où B_1^p est la boule unité ℓ_1 de $\mathbb{R}^p$. Cette quantité peut être interprétée comme le plus petit diamètre des sections de la boule unité ℓ_1 par des sous-espaces Γ_n de codimension au plus n. On rappelle que le diamètre d'une section est donnée par

$$\mathrm{diam}(\Gamma_n \cap B_1^p) = \sup_{x \in \Gamma_n \cap B_1^p} \|x\|_{\ell_2} = \sup_{x \in \Gamma_n} \frac{\|x\|_{\ell_2}}{\|x\|_{\ell_1}}.$$

On voit que ce diamètre compare la norme ℓ_2 et la norme ℓ_1 sur le sous-espace Γ_n, le rapport de ces deux normes est appelé la distorsion. Ce diamètre est intimement lié à la reconstruction exacte des vecteurs s-parcimonieux. De plus, la largeur de Gelfand permet d'évaluer le risque ℓ_2 de l'estimateur minimax étant donnée l'observation $y = X\beta^\star$. En particulier, on peut montrer [CDD09] que le basis pursuit a un risque ℓ_2 aussi faible (à une constante près) que l'estimateur minimax. On peut de même lier la constante de parcimonie à la largeur de Gelfand, comme le montre la proposition suivante.

Proposition I.6 — *Soit s un entier tel que $0 < s \leq p$. Si $X \in \mathbb{R}^{n\times p}$ est tel que*

$$\mathrm{diam}(\ker(X) \cap B_1^p) < \frac{1}{2\sqrt{s}},$$

alors X satisfait la propriété NSP d'ordre s. En particulier, la proposition I.5 montre que le basis pursuit (I.11) reconstruit exactement tous les vecteurs s-parcimonieux.

Démonstration. Soit $T \subseteq \{1, \dots, p\}$ de taille t et $h \in \ker(X)$, on a

$$\|h_T\|_{\ell_1} \leq \sqrt{t}\|h\|_{\ell_2} < \frac{1}{2}\Big(\frac{t}{s}\Big)^{1/2}\|h\|_{\ell_1}.$$

Ce qui conclut la preuve. □

Il a été prouvé [Kas77] qu'il existe des sous-espaces Γ_n de codimension n tels que

$$\mathrm{diam}(\Gamma_n \cap B_1^p) \leq C\left(\frac{1+\log(p/n)}{n}\right)^{1/2}, \tag{I.13}$$

où $C > 0$ est une constante universelle. Ainsi, la proposition I.6 montre qu'il existe des designs $X \in \mathbb{R}^{n\times p}$ pour lesquels tout vecteur $\beta^\star$ s-parcimonieux tel que $s \leq (\mathrm{cte}) \cdot n/\log(p/n)$ est l'unique solution du basis pursuit (I.11) étant donné l'observation $y = X\beta^\star$. Il suffit donc de

$$n \geq (\mathrm{cte}) \cdot s\log(p/s) \tag{I.14}$$

mesures pour parfaitement reconstruire n'importe quel vecteur de $\mathbb{R}^p$ ayant seulement s coefficients non nuls. Cette remarque est au cœur du Compressed Sensing [Dono6] et des remarquables applications qu'il apporte [DLP07]. En effet, tout vecteur parcimonieux $\beta^\star$ peut être reconstruit à l'aide d'un programme linéaire (et donc très efficace en terme de temps de calcul) à partir d'un nombre d'observations de l'ordre (à un facteur log près) de la complexité (i.e. la taille du support) de $\beta^\star$. L'approche classique du Compressed Sensing passe par la propriété d'isométrie restreinte RIP qui nous permet de donner des exemples de designs satisfaisant la borne (I.14).

2.4. La propriété d'isométrie restreinte. L'une des plus importantes conditions du modèle linéaire parcimonieux est sans aucun doute la propriété d'isométrie restreinte.

Définition I.3 (RIP(k, θ_k)) **—** *Une matrice $X \in \mathbb{R}^{n\times p}$ satisfait la propriété d'isométrie restreinte d'ordre k si et seulement si il existe $0 < \theta_k < 1$ (le plus petit possible) tel que*

$$\forall \gamma \in \mathbb{R}^p \quad k\text{-parcimonieux}, \quad (1-\theta_k)\|\gamma\|_{\ell_2}^2 \leq \|X\gamma\|_{\ell_2}^2 \leq (1+\theta_k)\|\gamma\|_{\ell_2}^2. \tag{I.15}$$

Cette condition est suffisante pour garantir que le basis pursuit reconstruit exactement tous les vecteurs s-parcimonieux.

Proposition I.7 — *Soit s un entier tel que $0 < s \leq p$. Soit $X \in \mathbb{R}^{n\times p}$ tel que X satisfait* RIP$(2s, \theta_{2s})$. *Soit $\beta^\star \in \mathbb{R}^p$ un vecteur s-parcimonieux.*

- (i) *Si $\theta_{2s} < 1$ alors $\beta^\star$ est l'unique solution de l'estimateur combinatoire (I.10),*
- (ii) *Si $\theta_{2s} < \sqrt{2} - 1$ alors la solution du basis pursuit (I.11) est unique et vaut $\beta^\star$.*

Démonstration. On commence par prouver (i). Soit γ un vecteur s-parcimonieux tel que $X\gamma = X\beta^\star$. On pose $h = \beta^\star - \gamma$, ce dernier à $\ker(X)$ et a au plus $2s$ coefficients non nuls. Comme $\theta_{2s} < 1$ et $Xh = 0$, il s'en suit que $h = 0$. Ainsi $\gamma = \beta^\star$, ce qui montre que $\beta^\star$ est l'unique vecteur s-parcimonieux ayant pour observation $y = X\beta^\star$ et donc l'unique solution de (I.10).

Passons à la preuve de (ii). On note T_0 le support de $\beta^\star$ et $h = \beta^{bp} - \beta^\star$. On a

$$(\|h_{T_0^c}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1}) + \|\beta^\star\|_{\ell_1} \leq \|\beta^\star + h\|_{\ell_1} \leq \|\beta^\star\|_{\ell_1},$$

où T_0^c est le complémentaire du sous-ensemble T_0. Ce qui donne

$$\|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_1}. \tag{I.16}$$

Décomposons l'ensemble T_0^c en sous-ensembles de cardinal s (sauf peut-être le dernier)

$$T_0^c = T_1 \cup T_2 \cup \ldots \cup T_l\,,$$

où T_1 est l'ensemble des indices des s plus grands coefficients (en valeur absolue) de $h_{T_0^c}$, T_2 est l'ensemble des indices des s plus grands coefficients de $h_{(T_0 \cup T_1)^c}$, et ainsi de suite. Pour $i \geq 2$, on remarque que

$$\|h_{T_i}\|_{\ell_2}^2 = \sum_{k \in T_i} h_k^2 \leq s^{-1} \|h_{T_{i-1}}\|_{\ell_1}^2\,,$$

en utilisant $|h_k| \leq s^{-1}\|h_{T_{i-1}}\|_{\ell_1}$. À l'aide de (I.16), il vient que

$$\sum_{i \geq 2} \|h_{T_i}\|_{\ell_2} \leq s^{-1/2} \sum_{i \geq 1} \|h_{T_i}\|_{\ell_1} = s^{-1/2} \|h_{T_0^c}\|_{\ell_1} \leq s^{-1/2} \|h_{T_0}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_2}. \tag{I.17}$$

De plus $Xh_{T_0 \cup T_1} = Xh - \sum_{i \geq 2} Xh_{T_i} = -\sum_{i \geq 2} Xh_{T_i}$, ce qui donne

$$(1 - \theta_{2s}) \|h_{T_0 \cup T_1}\|_{\ell_2}^2 \leq \|Xh_{T_0 \cup T_1}\|_{\ell_2}^2 = -\langle Xh_{T_0 \cup T_1}\,, \sum_{i \geq 2} Xh_{T_i} \rangle. \tag{I.18}$$

L'identité de polarisation montre que

$$\forall\, i \geq 2, \quad |\langle Xh_{T_0 \cup T_1}\,, Xh_{T_i} \rangle| \leq \sqrt{2}\,\theta_{2s} \|h_{T_0 \cup T_1}\|_{\ell_2} \|h_{T_i}\|_{\ell_2}.$$

Ainsi, en utilisant (I.17), la majoration (I.18) devient

$$(1 - \theta_{2s}) \|h_{T_0}\|_{\ell_2} \leq (1 - \theta_{2s}) \|h_{T_0 \cup T_1}\|_{\ell_2} \leq \sqrt{2}\,\theta_{2s} \sum_{i \geq 2} \|h_{T_i}\|_{\ell_2} \leq \sqrt{2}\,\theta_{2s} \|h_{T_0}\|_{\ell_2}.$$

Soit encore $(\sqrt{2} - 1 - \theta_{2s}) \|h_{T_0}\|_{\ell_2} \leq 0$. Comme $\theta_{2s} < \sqrt{2} - 1$, il vient que $\|h_{T_0}\|_{\ell_2} = 0$. La majoration (I.16) montre alors que $h = 0$, c'est-à-dire $\beta^{bp} = \beta^\star$. $\square$

Quelques exemples classiques de matrices vérifiant RIP sont présentés dans le tableau suivant (on conjecture que la borne $s \log(p/s)$ est vraie dans le cas Fourier).

Références	Échantillonnage	Nombre minimal d'observations n_0
[CRT06a]	Gaussien$^{(1)}$	$(\text{cte}) \cdot s \log(p/s)$
[CRT06a]	Bernoulli$^{(2)}$	$(\text{cte}) \cdot s \log(p/s)$
[RV08]	Fourier$^{(3)}$	$(\text{cte}) \cdot s \log^4(p)$

Table 1. Soit $X \in \mathbb{R}^{n \times p}$ tel que $^{(1)}$ les entrées $X_{i,j}$ sont i.i.d $\mathcal{N}(0, 1/\sqrt{n})$, $^{(2)}$ les entrées $X_{i,j}$ sont i.i.d selon une loi de Bernoulli de paramètre $1/2$, ou bien $^{(3)}$ dont les lignes sont tirées uniformément parmi les lignes de la matrice de Fourier. Si $n \geq n_0$ alors X satisfait RIP$(s, 1/3)$ avec grande probabilité.

Ainsi on peut reconstruire un vecteur s-parcimonieux à partir de $s \log(p/s)$ mesures (cas Gaussien ou Bernoulli). On a illustré le cas Gaussien avec une photo voir la figure 1.3.

Originale | Reconstruite

Figure 1.3. Sur la gauche, l'image originale de $p = 800,000$ pixels est une superposition de $s = 30,000$ ondelettes Daubechies-6. Sur la droite, la solution du basis pursuit à partir de $n = 100,000$ mesures Gaussiennes i.i.d. On observe qu'il y a reconstruction exacte de l'image originale.

Dans le cas où le vecteur cible $\beta^\star$ n'est pas parcimonieux, on s'intéresse à la « meilleure approximation de $\beta^\star$ par s termes ». Elle est représentée par le vecteur $\beta^\star_{\mathcal{S}_\star}$ où $\mathcal{S}_\star$ est l'ensemble des indices des s plus grands coefficients de $\beta^\star$. En particulier, dans le cas idéal où l'on connait à l'avance l'ensemble $\mathcal{S}_\star$, le meilleur estimateur à partir de s observations est tout simplement $\beta^\star_{\mathcal{S}_\star}$. Le théorème suivant montre que le basis pursuit est comparable à cet estimateur idéal.

Théorème I.8 ([CRT06b]) — *Soit s un entier tel que $0 < s \leq p$. Soit $X \in \mathbb{R}^{n\times p}$ tel que X satisfait $\mathrm{RIP}(2s, \theta_{2s})$ avec $\theta_{2s} < \sqrt{2} - 1$. Alors*

$$\|\beta^{bp} - \beta^\star\|_{\ell_1} \leq C \|\beta^\star - \beta^\star_{\mathcal{S}_\star}\|_{\ell_1},$$
$$\|\beta^{bp} - \beta^\star\|_{\ell_2} \leq C \frac{\|\beta^\star - \beta^\star_{\mathcal{S}_\star}\|_{\ell_1}}{\sqrt{s}},$$

où $C > 0$ est une constante qui dépend uniquement de θ_{2s}.

En particulier, avec seulement $s\log(p/s)$ observations (dans le cas d'un design Gaussien ou Bernoulli), le basis pursuit donne une estimation de $\beta^\star$ quasiment identique à la meilleure approximation par s termes. Le facteur log est alors le (faible) prix qu'il faut payer quand on ne connait pas l'ensemble $\mathcal{S}_\star$ des s plus grands coefficients de la cible. Il est possible de lier la propriété RIP avec le diamètre de la section de la boule unité ℓ_1 par le noyau du design. La proposition suivante nous a été montrée par G. Lécué.

Proposition I.9 — *Soit s un entier tel que $0 < s \leq p$. Si $X \in \mathbb{R}^{n\times p}$ satisfait $\mathrm{RIP}(8s, \theta_{8s})$ avec $\theta_{8s} < \sqrt{2} - 1$ alors*

$$\mathrm{diam}(\ker(X) \cap B_1^p) < \frac{1}{2\sqrt{s}}.$$

Démonstration. Soit $h \in \ker(X) \cap B_1^p$. On note T_0 l'ensemble des indices des $4s$ plus grands coefficients de h. On note T_0^c est le complémentaire de T_0. Décomposons l'ensemble T_0^c en sous-ensembles de cardinal $4s$ (sauf peut-être le dernier)

$$T_0^c = T_1 \cup T_2 \cup \ldots \cup T_l\,,$$

où T_1 est l'ensemble des indices des $4s$ plus grands coefficients (en valeur absolue) de $h_{T_0^c}$, T_2 est l'ensemble des indices des $4s$ plus grands coefficients de $h_{(T_0 \cup T_1)^c}$, et ainsi de suite. Il vient que

$$\|h_{(T_0\cup T_1)^c}\|_{\ell_2} \leq \sum_{i\geq 2} \|h_{T_i}\|_{\ell_2} \leq (4s)^{-1/2} \sum_{i\geq 1} \|h_{T_i}\|_{\ell_1} = (4s)^{-1/2} \|h_{T_0^c}\|_{\ell_1} \tag{I.19}$$

De plus $Xh_{T_0\cup T_1} = Xh - \sum_{i\geq 2} Xh_{T_i} = -\sum_{i\geq 2} Xh_{T_i}$, ce qui donne

$$(1-\theta_{8s})\|h_{T_0\cup T_1}\|_{\ell_2}^2 = \|Xh_{T_0\cup T_1}\|_{\ell_2}^2 = -\langle Xh_{T_0\cup T_1}\,,\, \sum_{i\geq 2} Xh_{T_i}\rangle. \tag{I.20}$$

L'identité de polarisation montre que

$$\forall\, i\geq 2, \quad |\langle Xh_{T_0\cup T_1}\,,\, Xh_{T_i}\rangle| \leq \sqrt{2}\,\theta_{8s}\|h_{T_0\cup T_1}\|_{\ell_2}\|h_{T_i}\|_{\ell_2}.$$

Ainsi, en utilisant (I.19), la majoration (I.20) devient

$$(1-\theta_{8s})\|h_{T_0\cup T_1}\|_{\ell_2} \leq \sqrt{2}\,\theta_{8s}\sum_{i\geq 2}\|h_{T_i}\|_{\ell_2} \leq \sqrt{2}\,\theta_{8s}\,(4s)^{-1/2}\|h_{T_0^c}\|_{\ell_1}. \tag{I.21}$$

Comme $\|h\|_{\ell_1}\leq 1$, (I.19) et (I.21) donnent

$$\|h\|_{\ell_2} \leq \|h_{(T_0\cup T_1)^c}\|_{\ell_2} + \|h_{T_0\cup T_1}\|_{\ell_2} \leq \Big(1+\frac{\sqrt{2}\,\theta_{8s}}{1-\theta_{8s}}\Big)\frac{1}{2\sqrt{s}} < \frac{1}{2\sqrt{s}},$$

en utilisant $\theta_{8s} < \sqrt{2}-1$. Ce qui conclut la démonstration. □

Ainsi on peut lier toutes les propriétés vues jusque là, cf Tableau 2.

$$\boxed{\mathrm{RIP}(8s,\theta_{8s}) \Longrightarrow \text{Gelfand} \Longrightarrow \mathrm{NSP}(s) \Longleftrightarrow \text{Reconstruction exacte}(s)}$$

Table 2. La propriété RIP implique Gelfand (i.e. borne sur le diamètre de la section de la boule unité ℓ_1 par le noyau du design, la proposition I.9) qui implique la propriété NSP (voir la proposition I.6). On note par NSP(s) la propriété NSP d'ordre s et par Reconstruction exacte(s) la reconstruction exacte de tous les vecteurs s-parcimonieux.

Dans la pratique, aucune des trois propriétés mentionnées ci-dessus n'est pas vérifiable. En particulier, même si l'on sait qu'avec grande probabilité les matrices Gaussiennes satisfont RIP, il est impossible de le vérifier pour une matrice donnée (aussi loin que l'on sache, ce problème est NP-difficile). C'est un vrai challenge pour la communauté statistique de trouver une condition garantissant de bonnes performances (i.e. proches de celle d'un estimateur idéal) tout en étant facilement vérifiable. Cette problématique est au cœur de cette thèse et nous nous efforcerons de trouver des designs déterministes (satisfaisant de facto la condition qui nous intéresse) ayant de bonnes performances.

3. Relaxation convexe en présence de bruit

Le basis pursuit est un excellent estimateur théorique qui nécessite des observations non bruitées. En pratique cette hypothèse n'est jamais satisfaite et l'on doit prendre en compte un bruit additionnel z non nul. On présente dans cette section deux estimateurs, le *lasso* et le *sélecteur Dantzig*. Ces deux estimateurs présentent les avantages du basis pursuit :

(1) ils peuvent être implémentés efficacement (en terme de temps de calcul),

(2) ils ont des « performances » proches d'un estimateur « idéal » (fourni par un oracle).

Bien entendu, leur analyse est plus délicate que celle du basis pursuit. Cependant notre but sera le même : *trouver des conditions sur la matrice de design pour garantir des performances proches de celle d'un estimateur idéal*. L'une d'entre elles est la propriété RIP mais nous verrons au cours de ce mémoire d'autres conditions comme la Restricted Eigenvalue Condition, la Compatibility Condition, la propriété UDP, etc ... Désormais **nous considérerons le modèle linéaire Gaussien** défini par (I.2).

3.1. Le lasso. Dans son article fondamental [Tib96] R. Tibshirani introduisit le lasso (Least Absolute Shrinkage and Selection Operator), il est défini par :

$$\beta^{\ell} \in \arg\min_{\beta\in\mathbb{R}^p} \left\{ \frac{1}{2}\|y - X\beta\|_{\ell_2}^2 + \lambda_\ell \, \|\beta\|_{\ell_1} \right\}, \tag{I.22}$$

où $\lambda_\ell > 0$ est un paramètre à régler. On commence par quelques commentaires :

- ✧ Ce programme minimise les moindres carrés $\|y - X\beta\|_{\ell_2}^2$ sous une pénalité ℓ_1. Cette dernière rend le lasso non linéaire par rapport à l'observation y. Ainsi, *il n'y a pas d'expression explicite* contrairement au cas de la régression ridge (I.7).
- ✧ De plus, on remarque que l'idée principale du lasso suit la propriété de « shrinkage » de la régression ridge. Plus précisément, le lasso contient une pénalité ℓ_1 sur les prédicteurs afin d'« écraser » les coefficients estimés vers 0 et, *in fine*, produire des solutions parcimonieuses.
- ✧ Le lasso est un SOCP (Second Order Cone Program) et il est possible de le calculer très efficacement.

Tout comme la régression ridge, le lasso peut être formulé en programmes équivalents, cf Figure 1.4.

Proposition I.10 — *Soit une observation $y \in \mathbb{R}^n$ et $\lambda_\ell > 0$. Soit β^ℓ une solution du lasso. On pose*

- ✦ $\mu_\ell = \|\beta^\ell\|_{\ell_1}$,
- ✦ $\nu_\ell = \|y - X\beta^\ell\|_{\ell_2}^2$.

Alors β^ℓ est aussi une solution des programmes suivants :

- ✦ $\beta^\ell \in \arg\min_{\|\beta\|_{\ell_1}\leq\mu_\ell} \|y - X\beta\|_{\ell_2}^2$,
- ✦ $\beta^\ell \in \arg\min_{\|y-X\beta\|_{\ell_2}^2\leq\nu_\ell} \|\beta\|_{\ell_1}$.

Le lasso est un estimateur précieux dans le modèle parcimonieux. En effet, et contrairement à la régression ridge, il favorise la parcimonie. Plus précisément, la pénalité ℓ_1 du lasso force l'estimateur à avoir beaucoup de coefficients nuls, cf Figure 1.5.

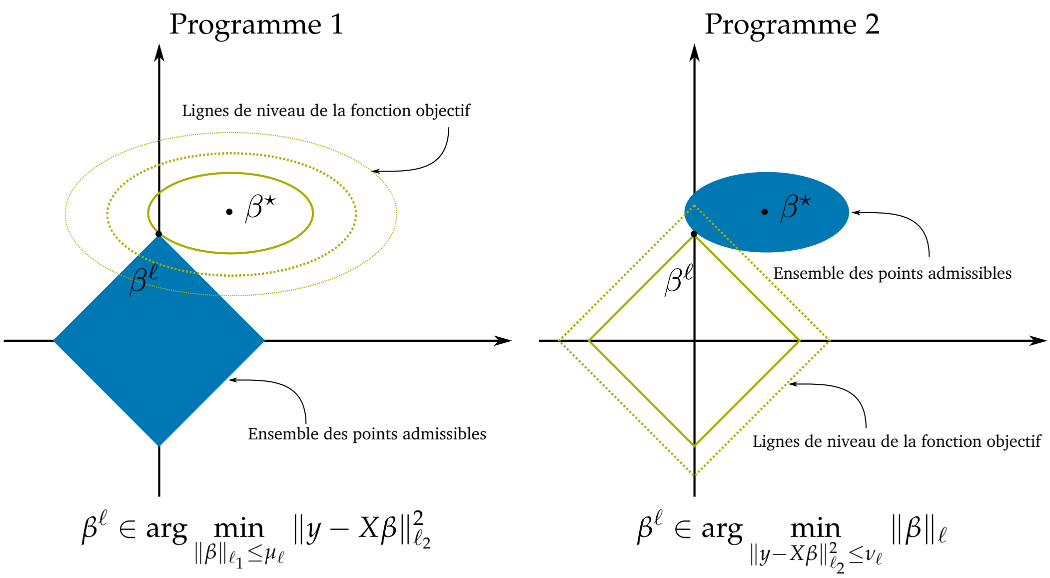


FIGURE 1.4. Les carrés représentent les lignes de niveau de la norme ℓ_1, alors que les ellipses celles de critère quadratique $\|y - X\beta\|_{\ell_2}^2$. Dans chaque programme, l'ensemble des points admissibles est peint en bleu. À gauche, les ellipses en pointillés représentent le programme 1 en train de minimiser les résidus quadratiques. À droite, le programme 2 minimise la norme ℓ_1. Quand $\mu_\ell = \|\beta^\ell\|_{\ell_1}$ et $\nu_\ell = \|y - X\beta^\ell\|_{\ell_2}^2$, ces deux programmes sont équivalents au lasso.

3.2. Le sélecteur Dantzig. Dans leur article [CT07] E.J. Candès et T. Tao introduisirent le *sélecteur Dantzig*. Il est défini par :

$$\beta^d \in \arg \min_{\beta \in \mathbb{R}^p} \|\beta\|_{\ell_1} \text{ sachant que } \|X^\top (y - X\beta)\|_{\ell_\infty} \leq \lambda_d \,, \tag{I.23}$$

où $\lambda_d > 0$ est un paramètre à régler. Le nom de cet estimateur rend hommage au père de la programmation linéaire, G.B. Dantzig [Cot06]. Soit $\beta \in \mathbb{R}^p$, on note $r = y - X\beta$ le vecteur des *résidus* au point β. L'ensemble des points admissibles du sélecteur Dantzig (I.23) est alors $\|X^\top r\|_{\ell_\infty} \leq \lambda_d$. On remarque que la contrainte porte sur la taille des résidus corrélés $X^\top r$ plutôt que sur la taille des résidus r eux-même. En s'appuyant sur [CT07], on justifie ce choix comme suit :

✧ On remarque que le sélecteur Dantzig est invariant par transformation orthonormale. Plus précisément, l'ensemble des points admissibles est invariant

$$(UX)^\top (UX\beta - Uy) = X^\top (X\beta - y) \,,$$

où $U \in \mathbb{R}^{n \times n}$ est tel que $U^\top U = \mathrm{Id}_n$. Cependant, si l'ensemble des points admissibles avait été défini par $\|r\|_{\ell_\infty} \leq \lambda_d$, alors l'estimateur n'aurait pas été invariant.

✧ Supposons que le résidu r soit égal à un prédicteur X_i et que $\|X_i\|_{\ell_\infty} \leq \lambda_d$. Évidement l'expérimentateur voudrait inclure ce prédicteur au modèle. On remarque que si l'ensemble des points admissibles avait été défini par $\|r\|_{\ell_\infty} \leq \lambda_d$, alors le résidu r aurait été admissible ce qui n'a aucun sens. Par contre, on remarque que $\|X^\top r\|_{\ell_\infty}$ est grand et donc r n'est pas admissible (pour des valeurs raisonnables du niveau de bruit). Le sélecteur Dantzig inclura à juste titre le prédicteur X_i au modèle.

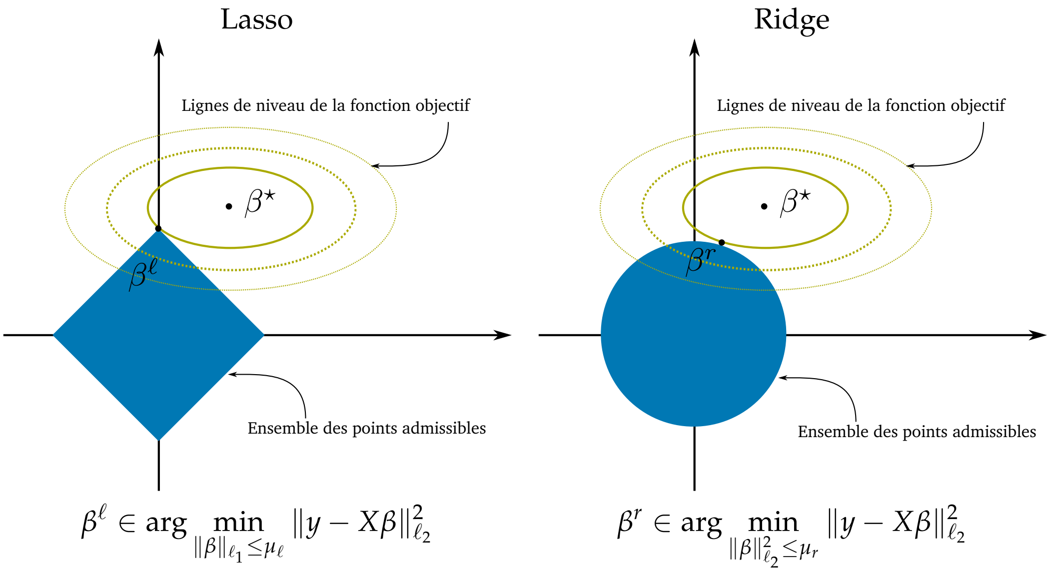


FIGURE 1.5. Le lasso produit beaucoup de coefficients nuls. Les zone bleues représentent l'ensemble des points admissibles tandis que les ellipses représentent les contours de l'erreur des moindres carrés. Le lasso β^ℓ est au point de contact d'une somme résiduelle lisse de carrés et d'une partie anguleuse de la boule ℓ_1. Ainsi, les ellipses seront plus facilement en contact avec une facette de faible dimension (i.e. un ensemble de vecteurs parcimonieux), et le point de contact aura beaucoup de coefficients nuls.

En conclusion, ce n'est pas la taille du résidu qui importe mais la taille des résidus corrélés. Le sélecteur Dantzig (I.23) est un programme convexe équivalent à un programme linéaire, à savoir

$$\min_{u,\beta\in\mathbb{R}^p} \sum_{i=1}^p u_i \quad \text{sachant que} \quad -u \leq \beta \leq u, \quad -\lambda_d \mathbf{1} \leq X^\top(y - X\beta) \leq \lambda_d \mathbf{1}\,,$$

où les variables d'optimisation sont $u, \beta \in \mathbb{R}^p$, et $\mathbf{1} \in \mathbb{R}^p$ est le vecteur dont tous les coefficients valent 1. À la page 6, nous avons vu qu'un tel programme peut être efficacement résolus sur ordinateur.

3.3. Des méthodes de seuillage doux. Pour des raisons pédagogiques, on s'intéresse au cas où le design est orthonormé et où $n \geq p$. Soit $X \in \mathbb{R}^{n\times p}$ une matrice de design telle que $X^\top X = \mathrm{Id}_p$. La solution des moindres carrés (I.3) est alors

$$\beta^{\ell s} = X^\top y.$$

Dans le cas orthonormé la solution du lasso et du sélecteur Dantzig est explicite, cf la proposition suivante. En particulier, celle-ci montre que ces estimateurs sont des seuillages doux, voir Figure 1.6.

Proposition I.11 — *Soit $X \in \mathbb{R}^{n\times p}$ tel que $X^\top X = \mathrm{Id}_p$. On note λ_ℓ (resp. λ_d et λ_r) le paramètre à régler du lasso (resp. sélecteur Dantzig et régression ridge). On suppose que*

$$\lambda_\ell = \lambda_d = \lambda_r = \lambda,$$

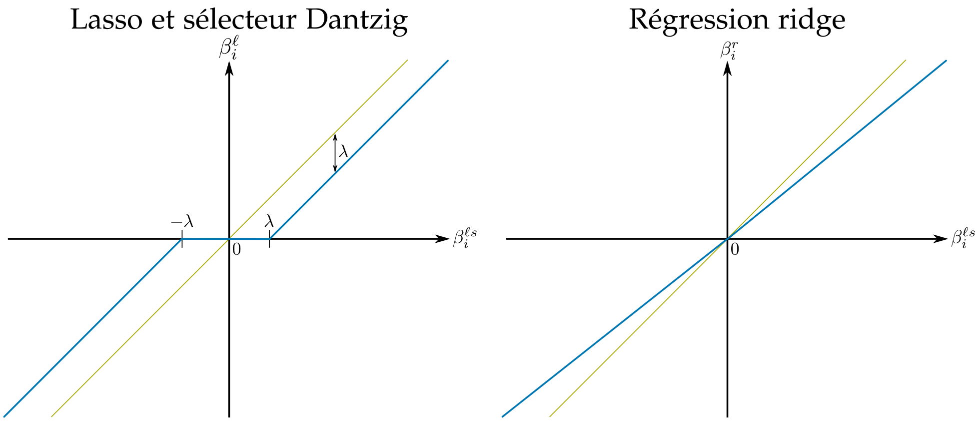


FIGURE 1.6. Le lasso et le sélecteur Dantzig sont des seuillages doux. Ils mettent à zéro tous les coefficients $\beta_i^{\ell s}$ (de la solution des moindres carrés) plus petit que le seuil λ, alors que la régression ridge les réduit mais ne les annule pas.

où $\lambda > 0$ est la valeur commune à ces trois paramètres. Soit une observation $y \in \mathbb{R}^n$, alors le lasso (resp. le sélecteur Dantzig) a une unique solution β^ℓ (resp. β^d) telle que

$$\beta_i^d = \beta_i^\ell = \operatorname{sgn}(\beta_i^{\ell s})(|\beta_i^{\ell s}| - \lambda)^+, \tag{I.24}$$

où $x^+ = \max(x, 0)$ pour tout $x \in \mathbb{R}$, et $\beta^{\ell s} = X^\top y$. La régression ridge satisfait

$$\beta_i^r = \frac{1}{1+\lambda}\,\beta_i^{\ell s}. \tag{I.25}$$

DÉMONSTRATION. L'expression (I.25) suit de (I.7). Dans le cas orthonormé, le sélecteur Dantzig s'écrit

$$\beta^d \in \arg\min_{\beta\in\mathbb{R}^p} \|\beta\|_{\ell_1} \ \text{ sachant que } \ \|\beta^{\ell s} - \beta\|_{\ell_\infty} \leq \lambda\,.$$

Il n'est pas difficile de voir que l'expression (I.24) suit. De même, le lasso s'écrit

$$\beta^\ell \in \arg\min_{\beta\in\mathbb{R}^p} \left\{\frac{1}{2}\|y - X\beta\|_{\ell_2}^2 + \lambda\|\beta\|_{\ell_1}\right\}.$$

On remarque que la fonction objectif est convexe. Par optimalité, un sous-gradient de la fonction objectif au point β^ℓ est nul, ce qui donne

$$X^\top X\beta^\ell - X^\top y + \lambda\,\partial_{\|\cdot\|_{\ell_1}}(\beta^\ell) = 0, \tag{I.26}$$

$$\beta^\ell - \beta^{\ell s} + \lambda\,\partial_{\|\cdot\|_{\ell_1}}(\beta^\ell) = 0, \tag{I.27}$$

où $\partial_{\|\cdot\|_{\ell_1}}(\beta^\ell)$ est un sous-gradient de la norme ℓ_1 au point β^ℓ. On a

$$\beta_i^\ell = \beta_i^{\ell s} - \lambda\,\partial_{\|\cdot\|_{\ell_1}}(\beta^\ell), \tag{I.28}$$

pour tout $i = 1, \ldots, p$. On remarque qu'il existe ρ_i tel que $|\rho_i| < 1$ et

$$\partial_{\|\cdot\|_{\ell_1}}(\beta^\ell)_i = \begin{cases} \operatorname{sgn}(\beta_i^\ell) & \text{if } \beta_i^\ell \neq 0, \\ \rho_i & \text{otherwise.} \end{cases} \tag{I.29}$$

En utilisant (I.28), on déduit que $\operatorname{sgn}(\beta_i^{\ell s})\beta_i^\ell = \left|\beta_i^\ell\right|$. Une partie de l'identité (I.24) suit. □

On remarque qu'au point β^ℓ le sous-gradient de la norme ℓ_1 doit être solution d'un système d'équations (voir (I.26) et (I.27) dans la preuve ci-dessus). D'autre part, à chaque indice i tel que $\beta_i^\ell = 0$ les coordonnées des sous-gradients au point β^ℓ peuvent prendre toutes les valeurs comprises entre -1 et 1. Ainsi, la parcimonie d'un vecteur γ offre d'autant plus de « libertés » à l'ensemble des sous-gradients au point γ. Plus précisément, les contraintes imposées au point β^ℓ sont d'autant plus réalisables que le vecteur est parcimonieux. En résumé, la non-différentiabilité de la norme ℓ_1 au point parcimonieux « crée » de la parcimonie.

3.4. Un exemple d'inégalité oracle. Dans ce dernier paragraphe on revient sur la notion d'inégalité oracle qui nous sera très utile par la suite.

Définition I.4 (Inégalité oracle) — *Une inégalité oracle lie les performances (par rapport à un risque) d'un estimateur réel à celles d'un estimateur idéal qui s'appuie sur l'information (qui n'est pas connu en pratique) donnée par un oracle (i.e. une aide omnisciente).*

On finit ce chapitre sur un exemple d'inégalités oracles pour le sélecteur Dantzig. Plaçons-nous dans le cadre de la grande dimension, i.e. $n \ll p$. Soit $\beta^\star \in \mathbb{R}^p$ un vecteur s-parcimonieux. Supposons qu'un oracle nous donne le support $\mathcal{S}_\star$ de $\beta^\star$ à l'avance. Le meilleur estimateur de $\beta^\star$ étant donné l'observation $y \in \mathbb{R}^n$ est alors l'estimateur des moindres carrés :

$$\beta^{ideal} \in \arg\min_{\beta\in\mathbb{R}^p} \|y - X\beta\|_{\ell_2}^2 \quad \text{sachant que} \quad \mathrm{supp}(\beta) \subseteq \mathcal{S}_\star\,. \tag{I.30}$$

En régressant y sur les variables dans $\mathcal{S}_\star$, on a

$$\beta^{ideal} = \big(X_{\mathcal{S}_\star}^\top X_{\mathcal{S}_\star}\big)^{-1} X_{\mathcal{S}_\star}^\top y.$$

L'erreur moyenne quadratique est alors

$$\mathbb{E}\big[\|\beta^\star - \beta^{ideal}\|_{\ell_2}^2\big] = \mathrm{Trace}\big(X_{\mathcal{S}_\star}^\top X_{\mathcal{S}_\star}\big)^{-1} \cdot \sigma_n^2.$$

Supposons que le design satisfait la propriété $\mathrm{RIP}(s,\theta_s)$, alors $X_{\mathcal{S}_\star}^\top X_{\mathcal{S}_\star}$ est proche de l'identité et l'estimateur idéal à un risque quadratique donné par :

$$\mathbb{E}\big[\|\beta^\star - \beta^{ideal}\|_{\ell_2}^2\big] \approx s \cdot \sigma_n^2.$$

Il semble impossible de retrouver un risque comparable sans connaître le support à l'avance. Pourtant, la proposition suivante montre que le sélecteur Dantzig atteint un risque du même ordre, à un facteur log près.

Proposition I.12 (Inégalité oracle) — *Soit s un entier tel que $0 < s \leq p$. Supposons que la matrice de design $X \in \mathbb{R}^{n\times p}$ ait tous ces vecteurs colonne de norme ℓ_2 unité et qu'elle satisfasse la propriété $\mathrm{RIP}(2s,\theta_{2s})$ avec $\theta_{2s} < \sqrt{2}-1$. Si $\lambda_d \geq (\mathrm{cte})\cdot\sigma_n \cdot 2\sqrt{\log p}$ alors*

$$\|\beta^d - \beta^\star\|_{\ell_2}^2 \leq (\mathrm{cte})\cdot \log p \cdot s \cdot \sigma_n^2\,,$$

avec une probabilité plus grande que $1 - 1/(p\sqrt{\pi\log p})$.

Démonstration. Le vecteur $\beta^\star$ est admissible si $\|X^\top z\|_{\ell_\infty} \leq \lambda_d$. Or

$$\forall\, i \in \{1,\ldots,p\}, \quad X_i^\top z \sim \mathcal{N}(0,\sigma_n^2)\,.$$

Ainsi, en majorant la probabilité d'une union par la somme des probabilités, on a

$$\mathbb{P}\big[\|X^\top z\|_{\ell_\infty} > \lambda_d\big] \leq 2p\,\mathbb{P}\Big[\mathcal{N}(0,1) > \frac{\lambda_d}{\sigma_n}\Big] \leq 2p\,\mathbb{P}\big[\mathcal{N}(0,1) > 2\sqrt{\log p}\,\big] \leq \frac{1}{p\sqrt{\pi\log p}}\,.$$

Plaçons-nous sur l'évènement $\{\|X^\top z\|_{\ell_\infty} \leq \lambda_d\}$ pour lequel le vecteur $\beta^\star$ est admissible. On note T_0 le support de $\beta^\star$ et $h = \beta^d - \beta^\star$. On commence par montrer que le vecteur h satisfait deux contraintes, cf Figure 1.7.

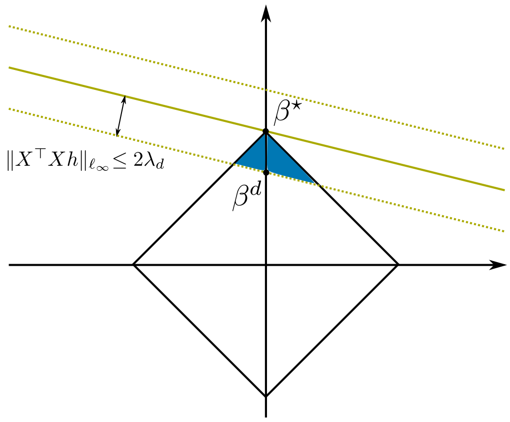


FIGURE 1.7. Le sélecteur Dantzig est sujet à deux contraintes dites respectivement du tube et du cône. La première montre qu'il ne peut pas être loin de l'espace affine $\{\beta \mid X\beta = X\beta^\star\}$, à savoir $\|X^\top Xh\|_{\ell_\infty} \leq 2\lambda_d$. La seconde montre que le sélecteur Dantzig appartient à la boule ℓ_1 de rayon $\|\beta^\star\|_{\ell_1}$. Finalement, il doit se situer dans le triangle bleu à l'intersection de ces deux contraintes.

✧ *Contrainte du tube :* $\|X^\top Xh\|_{\ell_\infty} \leq 2\lambda_d$. En effet, on a

$$\|X^\top Xh\|_{\ell_\infty} \leq \|X^\top(X\beta^d - y)\|_{\ell_\infty} + \|X^\top(X\beta^\star - y)\|_{\ell_\infty} \leq 2\lambda_d\,,$$

car $\beta^\star$ est admissible.

✧ *Contrainte du cône :* $\|h_{T_0^c}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_1}$. Comme $\beta^\star$ est admissible, il vient que

$$(\|h_{T_0^c}\|_{\ell_1} - \|h_{T_0}\|_{\ell_1}) + \|\beta^\star\|_{\ell_1} \leq \|\beta^\star + h\|_{\ell_1} \leq \|\beta^\star\|_{\ell_1},$$

ce qui démontre la contrainte du cône.

La preuve se déroule comme la preuve de la proposition I.7. Comme d'habitude, on décompose l'ensemble T_0^c en sous-ensembles de cardinal s (sauf peut-être le dernier)

$$T_0^c = T_1 \cup T_2 \cup \ldots \cup T_\ell\,,$$

où T_1 est l'ensemble des indices des s plus grands coefficients (en valeur absolue) de $h_{T_0^c}$, T_2 est l'ensemble des indices des s plus grands coefficients de $h_{(T_0 \cup T_1)^c}$, et ainsi de suite. En utilisant la contrainte du cône, on a

$$\|h_{(T_0\cup T_1)^c}\|_{\ell_2} \leq \sum_{i\geq 2} \|h_{T_i}\|_{\ell_2} \leq s^{-1/2} \sum_{i\geq 1} \|h_{T_i}\|_{\ell_1} = s^{-1/2}\|h_{T_0^c}\|_{\ell_1} \leq s^{-1/2}\|h_{T_0}\|_{\ell_1} \leq \|h_{T_0}\|_{\ell_2}. \tag{I.31}$$

De plus $Xh_{T_0\cup T_1} = Xh - \sum_{i\geq 2} Xh_{T_i}$, ce qui donne

$$(1-\theta_{2s})\|h_{T_0\cup T_1}\|_{\ell_2}^2 \leq \|Xh_{T_0\cup T_1}\|_{\ell_2}^2 = \langle Xh_{T_0\cup T_1}\,,\, Xh\rangle - \langle Xh_{T_0\cup T_1}\,,\, \sum_{i\geq 2} Xh_{T_i}\rangle. \tag{I.32}$$

On a déjà vu que l'identité de polarisation donne

$$\forall\, i \geq 2, \quad |\langle Xh_{T_0\cup T_1}\,,\, Xh_{T_i}\rangle| \leq \sqrt{2}\,\theta_{2s}\|h_{T_0\cup T_1}\|_{\ell_2}\|h_{T_i}\|_{\ell_2}.$$

ce qui mène à

$$|\langle Xh_{T_0\cup T_1}\,,\,\sum_{i\geq 2} Xh_{T_i}\rangle|\leq \sqrt{2}\,\theta_{2s}\|h_{T_0\cup T_1}\|_{\ell_2}\|h_{T_0}\|_{\ell_2}\,,$$

en utilisant (I.31). Le premier terme peut être majoré comme suit :

$$|\langle Xh_{T_0\cup T_1}\,,\,Xh\rangle|=|\langle h_{T_0\cup T_1}\,,\,X_{T_0\cup T_1}^{\top}Xh\rangle|\leq \|h_{T_0\cup T_1}\|_{\ell_2}\|X_{T_0\cup T_1}^{\top}Xh\|_{\ell_2}\leq \sqrt{2s}\lambda_d\,\|h_{T_0\cup T_1}\|_{\ell_2}\,,$$

en utilisant la contrainte du tube. Finalement, on obtient

$$(1-\theta_{2s})\|h_{T_0\cup T_1}\|_{\ell_2}\leq \sqrt{2}\,\theta_{2s}\|h_{T_0}\|_{\ell_2}+\sqrt{2s}\lambda_d\,.$$

Soit encore

$$\|h_{T_0\cup T_1}\|_{\ell_2}\leq \frac{\sqrt{2s}\lambda_d}{1-(1+\sqrt{2})\theta_{2s}}\,.$$

La contrainte du cône montre alors que

$$\|h\|_{\ell_2}\leq \frac{2\sqrt{2s}\lambda_d}{1-(1+\sqrt{2})\theta_{2s}}\,,$$

ce qui est le résultat attendu. $\square$

CHAPITRE II

La propriété universelle de distorsion

Numerous authors have established a connection between the Compressed Sensing problem without noise and the estimation of the Gelfand widths. This chapter shows that this connection is still true in the noisy case. Indeed, we investigate the lasso and the Dantzig selector in terms of the distortion of the design. This latter measures how far is the intersection between the kernel of the design matrix and the unit ℓ_1-ball from an ℓ_2-ball. In particular, we exhibit the weakest condition to get oracle inequalities in terms of the s-best term approximation. All the proofs can be found at the end of this chapter.

1. Oracle inequalities

Consider the high-dimensional linear model, c.f. (I.1). We recall that, in this model, an experimenter observes a vector $y \in \mathbb{R}^n$ such that

$$y = X\beta^\star + z,$$

where $X \in \mathbb{R}^{n\times p}$ denotes the design matrix (known from the experimenter), $\beta^\star \in \mathbb{R}^p$ is the target vector one would like to recover, and $z \in \mathbb{R}^n$ is a stochastic error term that contains all the perturbations of the experiment. Assume that one can provide a constant $\lambda^0 \in \mathbb{R}$ (wished as small as possible) such that

$$\|X^\top z\|_{\ell_\infty} \leq \lambda^0, \tag{II.1}$$

with an "overwhelming" probability. Observe that it is the only assumption on the noise throughout this chapter. This is a standard hypothesis and we recall a well-known result in the case where z is a n-multivariate Gaussian distribution.

Lemma II.1 — *Suppose that $z = (z_i)_{i=1}^n$ is such that the z_i's are i.i.d with respect to a Gaussian distribution with mean zero and variance σ_n^2. Choose $t \geq 1$ and set*

$$\lambda^0(t) = (1+t)\,\|X\|_{\ell_2,\infty} \cdot \sigma_n \cdot \sqrt{\log p},$$

where $\|X\|_{\ell_2,\infty}$ denotes the maximum ℓ_2-norm of the columns of X. Then,

$$\mathbb{P}\big\{\|X^\top z\|_{\ell_\infty} \leq \lambda^0(t)\big\} \geq 1 - \sqrt{2}/\Big[(1+t)\sqrt{\pi \log p}\; p^{\frac{(1+t)^2}{2}-1}\Big]. \tag{II.2}$$

Actually, our proof gives a better lower bound than (II.2), see (II.24) and the remark that follows.

As mentioned in Chapter I, a great statistical challenge is looking for efficiently verifiable conditions on X ensuring that the lasso (I.22) and the Dantzig selector (I.23) would recover "most of the information" about the target vector $\beta^\star$. We recall that the lasso is

$$\beta^\ell \in \arg\min_{\beta\in\mathbb{R}^p} \left\{ \frac{1}{2}\|y - X\beta\|_{\ell_2}^2 + \lambda_\ell\,\|\beta\|_{\ell_1} \right\}, \tag{I.22}$$

where $\lambda_\ell > 0$ is a tunning parameter, and the Dantzig selector is

$$\beta^d \in \arg\min_{\beta\in\mathbb{R}^p} \|\beta\|_{\ell_1} \ \text{ s.t. } \ \|X^\top(y - X\beta)\|_{\ell_\infty} \leq \lambda_d\,, \tag{I.23}$$

where $\lambda_d > 0$ is a tunning parameter. Some comments on these estimators can be found in Section 3, page 13.

What do we precisely mean by "most of the information" about the target? What is the amount of information one could recover from few observations? That are two of the important questions raised by Compressed Sensing. Suppose that you want to find an s-sparse vector that represents the target, then you would probably want that it contains the s largest (in magnitude) coefficients $\beta_i^\star$. More precisely, denote $\mathcal{S}_\star \subseteq \{1,\ldots,p\}$ the set of the indices of the s largest coefficients. The s-best term approximation vector is $\beta^\star_{\mathcal{S}_\star} \in \mathbb{R}^p$ where $(\beta^\star_{\mathcal{S}_\star})_i = \beta_i^\star$ if $i \in \mathcal{S}_\star$ and 0 otherwise (see Section 2.4, page 10). Observe that it is the s-sparse projection in respect to any ℓ_q-norm for $1 \leq q < +\infty$ (i.e. it minimizes the ℓ_q-distance to $\beta^\star$ among all the s-sparse vectors), and then the most natural approximation by an s-sparse vector.

Suppose that someone gives you all the keys to recover $\beta^\star_{\mathcal{S}_\star}$. More precisely, imagine that you know the subset $\mathcal{S}_\star$ a head of time in advance and that you observe

$$y^{oracle} = X\beta^\star_{\mathcal{S}_\star} + z\,.$$

This is an ideal situation called the oracle (the case of an s-sparse target was studied in (I.30)). Assume that the noise z is a Gaussian white noise of standard deviation σ_n, i.e. $z \sim \mathcal{N}_n(0, (\sigma_n^2)\,\mathrm{Id}_n)$ where $\mathcal{N}_n$ denotes the n-multivariate Gaussian distribution. Then the optimal estimator is the ordinary least square $\beta^{ideal} \in \mathbb{R}^p$ onto the subset $\mathcal{S}_\star$, namely

$$\beta^{ideal} \in \arg\min_{\substack{\beta\in\mathbb{R}^p\\ \mathrm{supp}(\beta)\subseteq\mathcal{S}_\star}} \|X\beta - y^{oracle}\|_{\ell_2}^2,$$

where $\mathrm{supp}(\beta) \subseteq \{1,\ldots,p\}$ denotes the support (i.e. the set of the indices of the non-zero coefficients) of the vector β. It holds

$$\|\beta^{ideal} - \beta^\star\|_{\ell_1} = \|\beta^{ideal} - \beta^\star_{\mathcal{S}_\star}\|_{\ell_1} + \|\beta^\star_{\mathcal{S}_\star^c}\|_{\ell_1} \leq \sqrt{s}\|\beta^{ideal} - \beta^\star_{\mathcal{S}_\star}\|_{\ell_2} + \|\beta^\star_{\mathcal{S}_\star^c}\|_{\ell_1},$$

where $\beta^\star_{\mathcal{S}_\star^c} = \beta^\star - \beta^\star_{\mathcal{S}_\star}$ denotes the ℓ_1-error of the s-best term approximation. An easy calculation shows that

$$\mathbb{E}\|\beta^{ideal} - \beta^\star_{\mathcal{S}_\star}\|_{\ell_2}^2 = \mathrm{Trace}\big((X_{\mathcal{S}_\star}^\top X_{\mathcal{S}_\star})^{-1}\big)\cdot\sigma_n^2 \geq \Big(\frac{1}{\rho_1}\Big)^2\cdot\sigma_n^2\cdot s,$$

where $X_{\mathcal{S}_\star} \in \mathbb{R}^{n\times s}$ denotes the matrix composed by the columns $X_i \in \mathbb{R}^n$ of the matrix X such that $i \in \mathcal{S}_\star$, and ρ_1 is the largest singular of X. It yields that

$$\Big[\mathbb{E}\|\beta^{ideal} - \beta^\star_{\mathcal{S}_\star}\|_{\ell_1}^2\Big]^{1/2} \geq \frac{1}{\rho_1}\cdot\sigma_n\cdot\sqrt{s}.$$

In a nutshell, the ℓ_1-distance between the target $\beta^\star$ and the optimal estimator β^{ideal} can be reasonably said of the order of

$$\frac{1}{\rho_1}\cdot\sigma_n\cdot s + \|\beta^\star_{\mathcal{S}_\star^c}\|_{\ell_1}. \tag{II.3}$$

We say that the lasso satisfies a *variable selection oracle inequality of order s* if and only if its ℓ_1-distance to the target, namely $\|\beta^\ell - \beta^\star\|_{\ell_1}$, is bounded by (II.3) up to a "satisfactory" multiplicative factor.

In some situations it could be interesting to have a good approximation of $X\beta^\star$. In the oracle case, we have

$$\|X\beta^{ideal} - X\beta^\star\|_{\ell_2} \leq \|X\beta^{ideal} - X\beta^\star_{\mathcal{S}_\star}\|_{\ell_2} + \|X\beta^\star_{\mathcal{S}^c_\star}\|_{\ell_2} \leq \|X\beta^{ideal} - X\beta^\star_{\mathcal{S}_\star}\|_{\ell_2} + \rho_1\|\beta^\star_{\mathcal{S}^c_\star}\|_{\ell_1}.$$

An easy calculation gives that

$$\mathbb{E}\|X\beta^{ideal} - X\beta^\star_{\mathcal{S}_\star}\|^2_{\ell_2} = \mathrm{Trace}\big(X_{\mathcal{S}_\star}\big(X^\top_{\mathcal{S}_\star}X_{\mathcal{S}_\star}\big)^{-1}X^\top_{\mathcal{S}_\star}\big)\cdot\sigma_n^2 = \sigma_n^2\cdot s.$$

Hence a tolerable upper bound is given by

$$\sigma_n\cdot\sqrt{s} + \rho_1\|\beta^\star_{\mathcal{S}^c_\star}\|_{\ell_1}. \tag{II.4}$$

We say that the lasso satisfies an *error prediction oracle inequality of order s* if and only if its prediction error is upper bounded by (II.4) up to a "satisfactory" multiplicative factor (say logarithmic in p).

2. The Universal Distortion Property

This chapter investigates a sufficient condition to prove oracle inequalities for the lasso, the Universal Distortion Property UDP(S_0, κ_0, Δ). This property can be verified using the notion of distortion.

2.1. The distortion. The distortion measures how far is the ℓ_1-norm from the Euclidean norm. It can be defined as follows.

Definition II.1 — *A subspace $\Gamma \subset \mathbb{R}^p$ has a distortion $\delta \in [1, \sqrt{p}]$ if and only if*

$$\forall x \in \Gamma, \quad \|x\|_{\ell_1} \leq \sqrt{p}\,\|x\|_{\ell_2} \leq \delta\|x\|_{\ell_1}.$$

A long standing issue in approximation theory in Banach spaces is finding "almost-Euclidean" sections of the unit ℓ_1-ball, i.e. subspaces with a distortion close to 1 and a dimension close to p. In particular, we recall that it has been established [Kas77] that there exists subspaces of dimension $p - n$ such that

$$\delta \leq C\left(\frac{p(1+\log(p/n))}{n}\right)^{1/2}, \tag{II.5}$$

where $C > 0$ is an universal constant, see (I.13) for more details. In other words, it was shown that, for all $n \leq p$, there exists a subspace Γ_n of dimension $p - n$ such that, for all $x \in \Gamma_n$,

$$\|x\|_{\ell_2} \leq C\left(\frac{1+\log(p/n)}{n}\right)^{1/2}\|x\|_{\ell_1}.$$

Recent deterministic constructions of almost-Euclidean sections of the ℓ_1-ball are presented in Section 4.

2.2. A verifiable condition. In the past decade, numerous conditions have been given to prove oracle inequalities for the lasso. An overview of some important conditions can be found in [BvdG09]. In this chapter, we are interested on two aspects of these conditions: Which one is the weakest ? Are they verifiable ? This latter means that one can tell if a given matrix satisfies the condition or not. As a matter of fact, answering this question is difficult. For instance, it is an open problem to find a computationally efficient algorithm that can tell if a given matrix satisfies the RIP condition or not. The UDP (see below) is the weakest condition to give oracle inequalities and a verifiable condition. More precisely, this condition is verifiable as soon as one can give an upper bound on the distortion of the kernel of the design matrix (see Lemma II.3).

Incidentally, it bends the community working on building almost-Euclidean subspaces and the statistics community.

Definition II.2 ([**dC11b**], UDP(S_0,κ_0,Δ)) — *A matrix $X\in\mathbb{R}^{n\times p}$ satisfies the universal distortion condition of order S_0, magnitude κ_0 and parameter Δ if and only if*

- ✦ $1\leq S_0\leq p$,
- ✦ $0<\kappa_0<1/2$,
- ✦ *and for all $\gamma\in\mathbb{R}^p$, for all integers $s\in\{1,\ldots,S_0\}$, for all subsets $\mathcal{S}\subseteq\{1,\ldots,p\}$ such that $|\mathcal{S}|=s$, it holds*

$$\|\gamma_{\mathcal{S}}\|_{\ell_1}\leq \Delta\sqrt{s}\,\|X\gamma\|_{\ell_2}+\kappa_0\|\gamma\|_{\ell_1}. \tag{II.6}$$

This property is similar to the Compatibility Condition of P. Büllmann and S.A. van de Geer [**BvdG09**] although it is weaker (see Section 3 for a comparison with the usual conditions). As a matter of fact, every matrix satisfies the UDP condition with explicit parameters in terms of the geometry (i.e. the distortion) of its kernel, cf. Lemma II.3. From this point of view, we can say that the UDP condition is a verifiable condition: could one compute the distortion of the kernel of the design then an UDP condition follows, see Lemma II.3 below.

2.3. Universality and optimal sparsity level. We call the property "Universal Distortion" indeed it is satisfied by all the full rank matrices (Universal) and the parameters S_0 and Δ can be expressed in terms of the distortion of the kernel Γ of X. As a matter of fact, one can prove a stronger result than the UDP condition, namely:

Lemma II.2 ([**dC11b**]) — *Let $X\in\mathbb{R}^{n\times p}$ be a full rank matrix. Denote δ the distortion of its kernel and ρ_n its smallest singular value. Then, for all $\gamma\in\mathbb{R}^p$,*

$$\|\gamma\|_{\ell_2}\leq \frac{\delta}{\sqrt{p}}\|\gamma\|_{\ell_1}+\frac{2\delta}{\rho_n}\|X\gamma\|_{\ell_2}.$$

Equivalently, we have

$$\mathcal{B}:=\{\gamma\in\mathbb{R}^p\ |\ (\delta/\sqrt{p})\|\gamma\|_{\ell_1}+(2\delta/\rho_n)\|X\gamma\|_{\ell_2}\leq 1\}\subset B_2^p,$$

where B_2^p denotes the Euclidean unit ball, see Figure 2.1.

This results implies that every full rank matrix satisfies the UDP condition with parameters described as follows.

Lemma II.3 ([**dC11b**]) — *Let $X\in\mathbb{R}^{n\times p}$ be a full rank matrix. Denote δ the distortion of its kernel and ρ_n its smallest singular value. Let $0<\kappa_0<1/2$ then X satisfies* UDP(S_0,κ_0,Δ) *where*

$$S_0=\Big(\frac{\kappa_0}{\delta}\Big)^2 p\quad\text{and}\quad \Delta=\frac{2\delta}{\rho_n}. \tag{II.7}$$

This lemma is sharp in the following sense. The parameter S_0 represents (see Theorem II.4) the maximum number of coefficients that can be recovered using lasso, we call it the *sparsity level*. It is known [**CDD09**] that the best bound one could expect is

$$S_{opt}\approx n/\log(p/n),$$

up to a multiplicative constant. In the case where (II.5) holds, the sparsity level satisfies

$$S_0\approx\kappa_0^2\,S_{opt}. \tag{II.8}$$

It shows that any design matrix with low distortion satisfies the UDP condition with an optimal sparsity level.

Figure 1

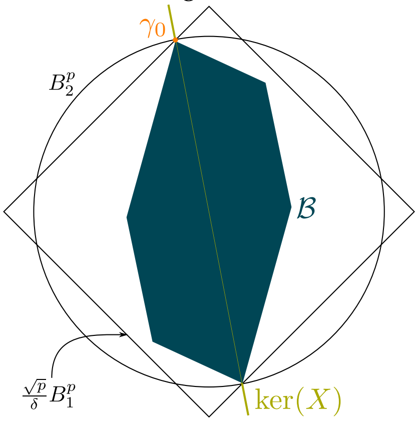


Figure 2

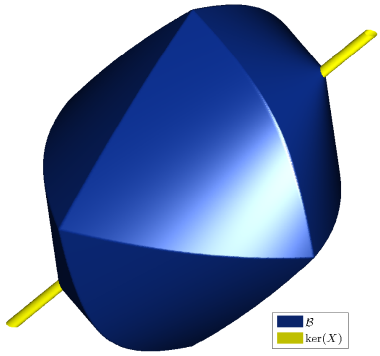


FIGURE 2.1. The set $\mathcal{B}$ is the unit ball of the norm defined by $\gamma \mapsto (\delta/\sqrt{p})\|\gamma\|_{\ell_1}+(2\delta/\rho_n)\|X\gamma\|_{\ell_2}$. One can see it as an interpolation between the standard ℓ_1-ball B_1^p and a tube $\|X\gamma\|_{\ell_2}\leq 1$. Lemma II.2 holds as soon as $\mathcal{B}$ is included in the Euclidean unit ball. Figure 1 depicts a ball $\mathcal{B}$ in two dimensions. In particular, $\mathcal{B}$ touches the intersection between the ℓ_1-sphere of radius $\sqrt{p}/\delta$ and the unit Euclidean sphere at a point γ_0 of $\ker(X)$. Figure 2 shows a ball $\mathcal{B}$ in three dimensions.

2.4. Results for the lasso. The results presented here fold into two parts. In the first part we assume that only UDP holds. In particular, it is not exclude that one can get better upper bounds on the parameters than Lemma II.3. As a matter of fact, the smaller Δ is the better the oracle inequalities are. In the second part, we give oracle inequalities in terms only of the distortion of the design.

Theorem II.4 ([dC11b]) — *Let $X \in \mathbb{R}^{n\times p}$ be a full rank matrix. Assume that X satisfies* UDP(S_0,κ_0,Δ) *and that $\|X^\top z\|_{\ell_\infty}\leq \lambda^0$. Then for any*

$$\lambda_\ell > \lambda^0/(1-2\kappa_0), \tag{II.9}$$

it holds

$$\|\beta^\ell-\beta^\star\|_{\ell_1}\leq \frac{2}{(1-\frac{\lambda^0}{\lambda_\ell})-2\kappa_0}\min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s,\ s\leq S_0.}}\Big(\lambda_\ell\,\Delta^2\,s+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big). \tag{II.10}$$

Using (II.7), the following holds: For every full rank matrix $X\in\mathbb{R}^{n\times p}$, for all $0<\kappa_0<1/2$ and λ_ℓ satisfying (II.9), we have

$$\|\beta^\ell-\beta^\star\|_{\ell_1}\leq \frac{2}{(1-\frac{\lambda^0}{\lambda_\ell})-2\kappa_0}\min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s,\\ s\leq(\kappa_0/\delta)^2p.}}\Big(\lambda_\ell\cdot\frac{4\,\delta^2}{\rho_n^2}\cdot s+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big), \tag{II.11}$$

where ρ_n denotes the smallest singular of X and δ the distortion of its kernel.

Remark — Consider the case where the noise satisfies the hypothesis of Lemma II.1 and take $\lambda^0=\lambda^0(1)$. Assume that κ_0 is constant (say $\kappa_0=1/3$) and take $\lambda_\ell=3\lambda^0$ then (II.10) becomes

$$\|\beta^\ell-\beta^\star\|_{\ell_1}\leq 12\min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s,\ s\leq S_0.}}\Big(6\,\|X\|_{\ell_2,\infty}\cdot\Delta^2\sqrt{\log p}\cdot\sigma_n\,s+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big),$$

which is an oracle inequality up to a multiplicative factor $\Delta^2\sqrt{\log p}$. In the same way, (II.11) becomes

$$\|\beta^\ell-\beta^\star\|_{\ell_1}\leq 12\min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\|\mathcal{S}|=s,\\ s\leq p/9\delta^2.}}\Big(24\,\|X\|_{\ell_2,\infty}\cdot\frac{\delta^2\sqrt{\log p}}{\rho_n}\cdot\frac{1}{\rho_n}\,\sigma_n\,s+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big),$$

which is an oracle inequality up to a multiplicative factor $C_{mult}:=(\delta^2\sqrt{\log p})/\rho_n$. If (II.5) holds, this latter becomes

$$C_{mult}=C\cdot\frac{p\,(1+\log(p/n))\,\sqrt{\log p}}{n\,\rho_n}, \tag{II.12}$$

where $C>0$ is the same universal constant as in (II.5). Roughly speaking, up to a factor of the order of (II.12), the lasso is as good as the oracle that knows the S_0-best term approximation of the target. Moreover, as mentioned in (II.8), S_0 is an optimal sparsity level. However, this multiplicative constant takes small values for a restrictive range of the parameter n. As a matter of fact, it is meaningful when n is a constant fraction of p.

Theorem II.5 ([dC11b]) — *Let $X\in\mathbb{R}^{n\times p}$ be a full rank matrix. Assume that X satisfies* $\mathrm{UDP}(S_0,\kappa_0,\Delta)$ *and that $\|X^\top z\|_{\ell_\infty}\leq\lambda^0$. Then for any*

$$\lambda_\ell>\lambda^0/(1-2\kappa_0), \tag{II.9}$$

it holds

$$\|X\beta^\ell-X\beta^\star\|_{\ell_2}\leq\min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\|\mathcal{S}|=s,\ s\leq S_0.}}\Big[4\lambda_\ell\,\Delta\sqrt{s}+\frac{\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}}{\Delta\sqrt{s}}\Big]. \tag{II.13}$$

Using (II.7), the following holds: For every full rank matrix $X\in\mathbb{R}^{n\times p}$, for all $0<\kappa_0<1/2$ and λ_ℓ satisfying (II.9), we have

$$\|X\beta^\ell-X\beta^\star\|_{\ell_2}\leq\min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\|\mathcal{S}|=s,\\ s\leq(\kappa_0/\delta)^2p.}}\Big[4\lambda_\ell\cdot\frac{2\,\delta}{\rho_n}\cdot\sqrt{s}+\frac{1}{2\delta\sqrt{s}}\cdot\rho_n\,\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big], \tag{II.14}$$

where ρ_n denotes the smallest singular of X and δ the distortion of its kernel.

Remark — Consider the case where the noise satisfies the hypothesis of Lemma II.1 and take $\lambda^0=\lambda^0(1)$. Assume that κ_0 is constant (say $\kappa_0=1/3$) and take $\lambda_\ell=3\lambda^0$ then (II.13) becomes

$$\|X\beta^\ell-X\beta^\star\|_{\ell_2}\leq\min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\|\mathcal{S}|=s,\ s\leq S_0.}}\Big[24\,\|X\|_{\ell_2,\infty}\cdot\Delta\sqrt{\log p}\cdot\sigma_n\,\sqrt{s}+\frac{\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}}{\Delta\sqrt{s}}\Big],$$

which is an oracle inequality up to a multiplicative factor of the order of $\Delta\sqrt{\log p}$. In the same way, (II.14) becomes

$$\|X\beta^\ell-X\beta^\star\|_{\ell_2}\leq\min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\},\\|\mathcal{S}|=s,\\ s\leq p/9\delta^2.}}\Big[48\,\|X\|_{\ell_2,\infty}\cdot\frac{\delta\sqrt{\log p}}{\rho_n}\cdot\frac{1}{\rho_n}\,\sigma_n\,\sqrt{s}+\frac{1}{2\delta\sqrt{s}}\cdot\rho_n\,\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big],$$

which is an oracle inequality up to a multiplicative factor $C'_{mult} := (\delta\sqrt{\log p})/\rho_n$. In the optimal case described by (II.5), this latter becomes

$$C'_{mult} = C \cdot \frac{(p\log p\,(1+\log(p/n)))^{1/2}}{\rho_n\sqrt{n}}, \tag{II.15}$$

where $C>0$ is the same universal constant as in (II.5).

2.5. Results for the Dantzig selector. Similarly, we derive the same results for the Dantzig selector. The only difference is that the parameter κ_0 must be less than $1/4$. Here again the results folds into two parts. In the first one, we only assume that UDP holds. In the second, we invoke Lemma II.3 to derive results in terms of the distortion of the design.

Theorem II.6 ([dC11b]) — *Let $X\in\mathbb{R}^{n\times p}$ be a full rank matrix. Assume that X satisfies* UDP(S_0,κ_0,Δ) *with $\kappa_0<1/4$ and that $\|X^\top z\|_{\ell_\infty}\leq\lambda^0$. Then for any*

$$\lambda_d > \lambda^0/(1-4\kappa_0), \tag{II.16}$$

it holds

$$\|\beta^d-\beta^\star\|_{\ell_1}\leq \frac{4}{(1-\frac{\lambda^0}{\lambda_d})-4\kappa_0}\min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s,\ s\leq S_0.}}\Big(\lambda_d\,\Delta^2\,s+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big). \tag{II.17}$$

Using (II.7), the following holds: For every full rank matrix $X\in\mathbb{R}^{n\times p}$, for all $0<\kappa_0<1/4$ and λ_d satisfying (II.16), we have

$$\|\beta^d-\beta^\star\|_{\ell_1}\leq \frac{4}{(1-\frac{\lambda^0}{\lambda_d})-4\kappa_0}\min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s,\\ s\leq(\kappa_0/\delta)^2p.}}\Big(\lambda_d\cdot\frac{4\,\delta^2}{\rho_n^2}\cdot s+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big), \tag{II.18}$$

where ρ_n denotes the smallest singular value of X and δ the distortion of its kernel.

The prediction error is given by the following theorem.

Theorem II.7 ([dC11b]) — *Let $X\in\mathbb{R}^{n\times p}$ be a full rank matrix. Assume that X satisfies* UDP(S_0,κ_0,Δ) *with $\kappa_0<1/4$ and that $\|X^\top z\|_{\ell_\infty}\leq\lambda^0$. Then for any*

$$\lambda_d > \lambda^0/(1-4\kappa_0), \tag{II.16}$$

it holds

$$\|X\beta^d-X\beta^\star\|_{\ell_2}\leq \min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s,\ s\leq S_0.}}\Big[4\lambda_d\,\Delta\sqrt{s}+\frac{\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}}{\Delta\sqrt{s}}\Big]. \tag{II.19}$$

Using (II.7), the following holds: For every full column rank matrix $X\in\mathbb{R}^{n\times p}$, for all $0<\kappa_0<1/4$ and λ_d satisfying (II.9), we have

$$\|X\beta^d-X\beta^\star\|_{\ell_2}\leq \min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s,\\ s\leq(\kappa_0/\delta)^2p.}}\Big[4\lambda_d\cdot\frac{2\,\delta}{\rho_n}\cdot\sqrt{s}+\frac{1}{2\delta\sqrt{s}}\cdot\rho_n\,\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big], \tag{II.20}$$

where ρ_n denotes the smallest singular of X and δ the distortion of its kernel.

Observe that the same comments as in the lasso case (e.g. (II.12), (II.15)) hold. Eventually, every result in constructing deterministic almost-Euclidean sections gives design that satisfies the oracle inequalities above.

3. An overview of the standards conditions

Oracle inequalities for the lasso have been established under a variety of different conditions on the design. An remarkable overview can be found in the article of P. Büllmann and S. A. van de Geer [BvdG09]. Though we recall some important sufficient conditions on the lasso and the Dantzig selector.

✦ **Restricted Isometry Property [CRT06b]:** The definition can be found in (I.15).

✦ **Restricted Eigenvalue Condition [BRT09]:** $X \in \mathbb{R}^{n\times p}$ satisfies $RE(S, c_0)$ if and only if

$$\kappa(S,c_0) = \min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\}\\ |\mathcal{S}|\leq S}} \min_{\substack{\gamma\neq 0\\ \|\gamma_{\mathcal{S}^c}\|_{\ell_1}\leq c_0\|\gamma_{\mathcal{S}}\|_{\ell_1}}} \frac{\|X\gamma\|_{\ell_2}}{\|\gamma_{\mathcal{S}}\|_{\ell_2}} > 0\,.$$

The constant $\kappa(S, c_0)$ is called the (S, c_0)-restricted ℓ_2-eigenvalue.

✦ **Compatibility Condition [BvdG09]:** $X \in \mathbb{R}^{n\times p}$ satisfies $Compatibility(S, c_0)$ if and only if

$$\phi(S,c_0) = \min_{\substack{\mathcal{S}\subseteq\{1,\dots,p\}\\ |\mathcal{S}|\leq S}} \min_{\substack{\gamma\neq 0\\ \|\gamma_{\mathcal{S}^c}\|_{\ell_1}\leq c_0\|\gamma_{\mathcal{S}}\|_{\ell_1}}} \frac{\sqrt{|\mathcal{S}|}\|X\gamma\|_{\ell_2}}{\|\gamma_{\mathcal{S}}\|_{\ell_1}} > 0\,.$$

The constant $\phi(S, c_0)$ is called the (S, c_0)-restricted ℓ_1-eigenvalue.

✦ **$\mathbf{H}_{S,1}$ Condition [JN11]:** $X \in \mathbb{R}^{n\times p}$ satisfies the $\mathbf{H}_{S,1}(\kappa)$ condition (with $\kappa < 1/2$) if and only if for all $\gamma \in \mathbb{R}^p$ and for all $\mathcal{S} \subseteq \{1,\dots,p\}$ such that $|\mathcal{S}| \leq S$, it holds

$$\|\gamma_{\mathcal{S}}\|_{\ell_1} \leq \hat\lambda\, S\, \|X\gamma\|_{\ell_2} + \kappa\|\gamma\|_{\ell_1},$$

where $\hat\lambda$ denotes the maximum of the ℓ_2-norms of the columns in X.

Remark — This latter condition is weaker than the UDP condition nevertheless the authors [JN11] established limits of performance on their conditions: the condition $\mathbf{H}_{s,\infty}(1/3)$ (that implies $\mathbf{H}_{s,1}(1/3)$) is feasible only in a severe restricted range of the sparsity parameter s. Notice that this is not the case of the UDP condition, the equality (II.8) shows that it is feasible for a large range of the sparsity parameter s (indeed an optimal range, cf. (II.8)).

Let us emphasize that the above description is not meant to be exhaustive. In particular we do not mention the irrepresentable condition [YZ06] which ensures exact recovery of the support. The next proposition shows that the UDP condition is weaker than RIP, RE and Compatibility condition.

Proposition II.8 ([dC11b]) **—** *Let $X \in \mathbb{R}^{n\times p}$ be a full rank matrix, then the following is true:*

✦ *The* RIP$(5S, \theta_{5S})$ *condition with* $\theta_{5S} < \sqrt{2}-1$ *implies* UDP(S, κ_0, Δ) *for all pairs* (κ_0, Δ) *such that*

$$\left(1+2\Big(\frac{1-\theta_{5S}}{1+\theta_{5S}}\Big)^{1/2}\right)^{-1} < \kappa_0 < 0.5 \text{ and } \Delta \geq \left(\sqrt{1-\theta_{5S}} + \frac{\kappa_0-1}{2\kappa_0}\sqrt{1+\theta_{5S}}\right)^{-1}. \quad \text{(II.21)}$$

✦ *The $RE(S, c_0)$ condition implies* UDP$(S, c_0, \kappa(S, c_0)^{-1})$.

✦ *The $Compatibility(S, c_0)$ condition implies* UDP$(S, c_0, \phi(S, c_0)^{-1})$.

Proof. It is obvious that $RE(S, c_0)$ condition implies UDP$(S, c_0, \kappa(S, c_0)^{-1})$, and that $Compatibility(S, c_0)$ condition implies UDP$(S, c_0, \phi(S, c_0)^{-1})$.

Assume that X satisfies RIP$(5S, \theta_{5S})$. Let $\gamma \in \mathbb{R}^p$, $s \in \{1, \ldots, S_0\}$, and $T_0 \subseteq \{1, \ldots, p\}$ such that $|T_0| = s$. Choose a pair (κ_0, Δ) as in (II.21).

✧ If $\|\gamma_{T_0}\|_{\ell_1} \leq \kappa_0 \|\gamma\|_{\ell_1}$ then $\|\gamma_{T_0}\|_{\ell_1} \leq \Delta\sqrt{s}\|X\gamma\|_{\ell_2} + \kappa_0\|\gamma\|_{\ell_1}$.

✧ Suppose that $\|\gamma_{T_0}\|_{\ell_1} > \kappa_0 \|\gamma\|_{\ell_1}$ then

$$\|\gamma_{T_0^c}\|_{\ell_1} < \frac{1-\kappa_0}{\kappa_0}\|\gamma_{T_0}\|_{\ell_1} . \tag{II.22}$$

Denote T_1 the set of the indices of the $4s$ largest coefficients (in absolute value) in T_0^c, denote T_2 the set of the indices of the $4s$ largest coefficients in $(T_0 \cup T_1)^c$, etc... Hence we decompose T_0^c into disjoints sets

$$T_0^c = T_1 \cup T_2 \cup \ldots \cup T_l .$$

Using (II.22), it yields

$$\sum_{i \geq 2}\|\gamma_{T_i}\|_{\ell_2} \leq (4s)^{-1/2}\sum_{i\geq 1}\|\gamma_{T_i}\|_{\ell_1} = (4s)^{-1/2}\|\gamma_{T_0^c}\|_{\ell_1} \leq \frac{1-\kappa_0}{2\kappa_0\sqrt{s}}\|\gamma_{T_0}\|_{\ell_1} \tag{II.23}$$

Using RIP$(5S, \theta_{5S})$ and (II.23), it follows that

$$\begin{aligned}
\|X\gamma\|_{\ell_2} &\geq \|X(\gamma_{(T_0\cup T_1)})\|_{\ell_2} - \sum_{i\geq 2}\|X(\gamma_{T_i})\|_{\ell_2} ,\\
&\geq \sqrt{1-\theta_{5S}}\,\|\gamma_{(T_0\cup T_1)}\|_{\ell_2} - \sqrt{1+\theta_{5S}}\sum_{i\geq 2}\|\gamma_{T_i}\|_{\ell_2} ,\\
&\geq \sqrt{1-\theta_{5S}}\,\|\gamma_{T_0}\|_{\ell_2} - \sqrt{1+\theta_{5S}}\,\frac{1-\kappa_0}{2\kappa_0}\,\frac{\|\gamma_{T_0}\|_{\ell_1}}{\sqrt{s}} ,\\
&\geq \Big(\sqrt{1-\theta_{5S}} + \frac{\kappa_0-1}{2\kappa_0}\sqrt{1+\theta_{5S}}\Big)\frac{\|\gamma_{T_0}\|_{\ell_1}}{\sqrt{s}} ,\\
&= \frac{\sqrt{1+\theta_{5S}}}{2\kappa_0}\bigg(1+2\Big(\frac{1-\theta_{5S}}{1+\theta_{5S}}\Big)^{1/2}\bigg)\bigg[\kappa_0 - \bigg(1+2\Big(\frac{1-\theta_{5S}}{1+\theta_{5S}}\Big)^{1/2}\bigg)^{-1}\bigg]\frac{\|\gamma_{T_0}\|_{\ell_1}}{\sqrt{s}} .
\end{aligned}$$

The lower bound on κ_0 shows that the right hand side is positive. This latter is exactly $\|\gamma_{T_0}\|_{\ell_1}/(\Delta\sqrt{s})$. Eventually, we get

$$\|\gamma_{T_0}\|_{\ell_1} \leq \Delta\sqrt{s}\|X\gamma\|_{\ell_2} \leq \Delta\sqrt{s}\|X\gamma\|_{\ell_2} + \kappa_0\|\gamma\|_{\ell_1} .$$

This ends the proofs. □

Table 1 summarizes the relation between the conditions for the lasso.

RIP $\Longrightarrow$ *REC* $\Longrightarrow$ *Compatibility* $\Longrightarrow$ UDP

Table 1. The RIP property implies REC that implies the Compatibility condition (see [BvdG09]) that implies UDP (see Proposition II.8).

The UDP condition is the weakest condition among all the condition on the lasso. As a matter of fact, the next section shows that one can construct (deterministic) subspaces with prescribed distortion. Hence it is possible to construct (deterministic) design matrices that satisfies the UDP condition.

4. The distortion of the design

One of the big issue in modern statistics is to find verifiable conditions. This question is valuable to the statistics community since one knows that the RIP condition (which is the key stone of Compressed Sensing) cannot be computationally checked for a given matrix. To overcome this difficulty, at the price of weaker results, we investigate the role of the distortion in high-dimensional regression. It is known that there is a connection between the Compressed Sensing problem and the problem of estimating the distortion. This framework was studied by numerous authors [CDD09, KT07, DeV07] and might interest both people working on building deterministic almost-Euclidean section of the ℓ_1-ball and those looking for deterministic design for Compressed Sensing.

4.1. Explicit constructions. Table 2 presents some important results for constructing almost-Euclidean sections of the ℓ_1-ball. The last line of Table 2 deals with the optimal case derived from a probabilistic construction. Even though this construction has been established in the late seventies there is no deterministic proof of it.

Reference	Distortion	Co-dimension	Randomness
[Ind07]	$1+\varepsilon$	$p-p^{1-o_\varepsilon(1)}$	Explicit
[GLR08]	$\log(p)^{O_\eta(\log\log\log p)}$	ηp	Explicit
[IS10]	$1+\varepsilon$	$(1-(\gamma\varepsilon)^{O(1/\gamma)})p$	$O(p^\gamma)$
[Kas77]	$C\,(p(1+\log(p/n))/n)^{1/2}$	n	np

TABLE 2. The best known results for constructing almost-Euclidean subspaces (see [IS10]). The parameters $\varepsilon, \eta, \gamma \in (0,1)$ are assumed to be constants, although we explicitly point out when the dependence on them is subsumed by the big-Oh notation. The parameter $C > 0$ denotes an universal constant. The last column gives the number of random bits which are necessary for each construction. The first part of table presents explicit constructions while the second one gives some important random constructions.

Most of the explicit constructions can be viewed as related to the context of error-correcting codes. As a matter of fact, the construction of [Ind07] is based on amplifying the minimum distance of a code using expanders. While the construction of [GLR08] is based on Low-Density Parity Check (LDPC) codes. Lastly, the construction of [IS10] is related to the tensor product of error-correcting codes. The main reason of this state of affairs is that the vectors of a subspace of low distortion must be "well-spread", i.e. a small subset of tis coordinates cannot contain most of its ℓ_2-norm (cf [Ind07, GLR08]). This property is required from a good error-correcting code, where the weight (i.e. the ℓ_0-norm) of each codeword cannot be concentrated on a small subset of its coordinates. As a matter of fact, this property can be seen as the NSP property (see Definition I.2, page 7) in Statistics.

4.2. Duality between Compressed Sensing and error correcting codes. There is a duality between Compressed Sensing matrices and the error correcting codes matrices. As a matter of fact, let p and $1 \le n < p$ be two integers. Let A be a $n \times p$ matrix and B

be a $p \times (p-n)$ matrix such that the $p \times p$ matrix

$$\begin{bmatrix} A \\ \hline B^\top \end{bmatrix} \in \mathcal{O}(\mathbb{R}^p)\,,$$

where $\mathcal{O}(\mathbb{R}^p)$ denotes the unitary group. For instance, if we are given an orthonormal basis $(\Phi_1, \ldots, \Phi_p)$ of $\mathbb{R}^p$, A can be constructed such that its n rows vectors have been selected (for instance at random without replacement) among $\{\Phi_1, \ldots, \Phi_p\}$ and that the $p-n$ rows vectors of $B^\top$ are the remaining orthonormal vectors of $\{\Phi_1, \ldots, \Phi_p\}$ which have not been selected for the construction of A. Under this assumption there are strong connections between the Compressed Sensing problem where A is used as a design matrix, and the error correcting codes where B is used as a coding matrix. It is in particular interesting to note that

$$AB = 0$$

thus A is a "parity-check matrix" for B. But we have more than that since $\mathrm{Range}(B) = \mathrm{Ker}(A)$. Since every code word belongs to its kernel, the matrix A can be used to decode any corrupted code word. It would essentially capture the "errors" (i.e. differences between the true word and the received word). In a nutshell, the design matrices of Compressed Sensing can be viewed as parity-check matrices for error-correcting codes. Hence every explicit construction of this latter gives explicit construction of design matrices.

5. Proofs

This section is devoted to the proof of the different results.

Proof of Lemma II.1 — Observe that $X^\top z \sim \mathcal{N}_p(0, \sigma_n^2 X^\top X)$. Hence,

$$\forall j = 1, \ldots, p, \quad X_j^\top z \sim \mathcal{N}\big(0, \sigma_n^2 \, \|X_j\|_{\ell_2}^2\big).$$

Using Šidák's inequality [Š68], it yields

$$\begin{aligned}\mathbb{P}\big(\|X^\top z\|_{\ell_\infty} \leq \lambda^0\big) &\geq \mathbb{P}\big(\|\tilde{z}\|_{\ell_\infty} \leq \lambda^0\big) \\ &= \prod_{i=1}^p \mathbb{P}\left(|\tilde{z}_i| \leq \lambda^0\right)\,,\end{aligned}$$

where the $\tilde{z}_i$'s are i.i.d. with respect to $\mathcal{N}\big(0, \sigma_n^2 \, \|X\|_{\ell_2,\infty}^2\big)$. Denote Φ and φ respectively the cumulative distribution function and the probability density function of the standard normal. Set $\theta = (1+t)\sqrt{\log p}$. It holds

$$\begin{aligned}\prod_{i=1}^p \mathbb{P}\left(|\tilde{z}_i| \leq \lambda^0\right) &= \mathbb{P}\left(|z_1| \leq \lambda^0\right)^p \\ &= (2\Phi(\theta) - 1)^p \\ &> \big(1 - 2\varphi(\theta)/\theta\big)^p\,,\end{aligned}$$

using an integration by parts to get $1 - \Phi(\theta) < \varphi(\theta)/\theta$. It yields that

$$\begin{aligned}\mathbb{P}\big(\|X^\top z\|_{\ell_\infty} \leq \lambda^0\big) &\geq \big(1 - 2\varphi(\theta)/\theta\big)^p\,, \qquad \text{(II.24)} \\ &\geq 1 - 2p\frac{\varphi(\theta)}{\theta}\,,\end{aligned}$$

$$= 1 - \frac{\sqrt{2}}{(1+t)\sqrt{\pi \log p}\; p^{\frac{(1+t)^2}{2}-1}} \,. \tag{II.2}$$

This concludes the proof. □

Remark — As a matter of fact, we can improve the bound (II.2). Indeed, the lower bound (II.24) is much better. For instance, for $p = 10^3$ and $t = 1$ (which can illustrate the case of an audio signal) it holds

$$\big(1 - 2\varphi(\theta)/\theta\big)^p \simeq 1 - 5 \times 10^{-6}\,,$$
$$\text{while} \quad 1 - 2p\frac{\varphi(\theta)}{\theta} \simeq 1 - 2 \times 10^{-4}\,.$$

Alike, when $p = 10^7$ and $t = 1$ (which can illustrate the case of a ten mega-pixel picture), we have

$$\big(1 - 2\varphi(\theta)/\theta\big)^p \simeq 1 - 7 \times 10^{-14}\,,$$
$$\text{while} \quad 1 - 2p\frac{\varphi(\theta)}{\theta} \simeq 1 - 1 \times 10^{-8}\,.$$

For sake of readability, we choose the bound (II.2) in the formulation of Lemma II.1. But we stress out that we can significantly improve it considering the bound (II.24).
— Last but not least, we underline that the Šidák's inequality allows us to deal with correlated noise.

Proof of Lemma II.2 — Consider the following singular value decomposition $X = U^\top D A$ where

- ✧ $U \in \mathbb{R}^{n\times n}$ is such that $UU^\top = \mathrm{Id}_n$,
- ✧ $D = \mathrm{Diag}(\rho_1, \ldots, \rho_n)$ is a diagonal matrix where $\rho_1 \geq \cdots \geq \rho_n > 0$ are the singular values of X,
- ✧ and $A \in \mathbb{R}^{n\times p}$ is such that $AA^\top = \mathrm{Id}_n$.

We recall that the only assumption on the design is that it has full column rank which yields that $\rho_n > 0$. Let δ be the distortion of the kernel Γ of the design. Denote by π_Γ (resp. $\pi_{\Gamma^\perp}$) the ℓ_2-projection onto Γ (resp. $\Gamma^\perp$). Let $\gamma \in \mathbb{R}^p$, then

$$\gamma = \pi_\Gamma(\gamma) + \pi_{\Gamma^\perp}(\gamma)\,.$$

In this decomposition the ℓ_2-norm of the first term is upper bounder by the distortion times the ℓ_1-norm. Moreover, an easy calculation shows that

$$\pi_{\Gamma^\perp}(\gamma) = A^\top A\gamma.$$

So we are able to upper bound, as follows:

$$\begin{aligned}
\|\gamma\|_{\ell_2} &\leq \|\pi_\Gamma(\gamma)\|_{\ell_2} + \|\pi_{\Gamma^\perp}(\gamma)\|_{\ell_2},\\
&\leq \frac{\delta}{\sqrt{p}}\,\|\pi_\Gamma(\gamma)\|_{\ell_1} + \|A^\top A\gamma\|_{\ell_2},\\
&\leq \frac{\delta}{\sqrt{p}}\,\big(\|\gamma\|_{\ell_1} + \|(\pi_{\Gamma^\perp}(\gamma))\|_{\ell_1}\big) + \|A\gamma\|_{\ell_2},\\
&\leq \frac{\delta}{\sqrt{p}}\,\|\gamma\|_{\ell_1} + \delta\,\|A^\top A\gamma\|_{\ell_2} + \|A\gamma\|_{\ell_2},\\
&\leq \frac{\delta}{\sqrt{p}}\,\|\gamma\|_{\ell_1} + (1+\delta)\,\|A\gamma\|_{\ell_2},
\end{aligned}$$

$$\leq \frac{\delta}{\sqrt{p}}\,\|\gamma\|_{\ell_1}+\frac{1+\delta}{\rho_n}\,\|X\gamma\|_{\ell_2},$$
$$\leq \frac{\delta}{\sqrt{p}}\,\|\gamma\|_{\ell_1}+\frac{2\delta}{\rho_n}\,\|X\gamma\|_{\ell_2},$$

using the triangular inequality and the distortion of the kernel Γ. This concludes the proof. □

Proof of Lemma II.3 — Let $s \in \{1,\dots,S\}$ and let $\mathcal{S} \subseteq \{1,\dots,p\}$ be such that $|\mathcal{S}| = s$. It holds,

$$\|\gamma_{\mathcal{S}}\|_{\ell_1} \leq \sqrt{s}\|\gamma\|_{\ell_2}$$

Using Lemma II.2, it follows that

$$\|\gamma_{\mathcal{S}}\|_{\ell_1} \leq \frac{\sqrt{s}}{\sqrt{p}}\,\delta\,\|\gamma\|_{\ell_1}+\frac{2\delta}{\rho_n}\,\sqrt{s}\,\|X\gamma\|_{\ell_2},$$

Eventually, set

$$\kappa_0 = (\sqrt{S}/\sqrt{p})\,\delta \quad \text{and} \quad \Delta = 2\delta/\rho_n\,.$$

This ends the proof. □

Proof of Theorem II.4 — We recall that λ^0 denotes an upper bound on the amplification of the noise, see (II.1). We begin with a standard result.

Lemma II.9 — *Let $h = \beta^\ell - \beta^\star \in \mathbb{R}^p$ and $\lambda_\ell \geq \lambda^0$. Then, for all subsets $\mathcal{S} \subseteq \{1,\dots,p\}$, it holds,*

$$\frac{1}{2\lambda_\ell}\Big[\frac{1}{2}\|Xh\|_{\ell_2}^2+(\lambda_\ell-\lambda^0)\|h\|_{\ell_1}\Big] \leq \|h_{\mathcal{S}}\|_{\ell_1}+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}. \tag{II.25}$$

Proof. By optimality, we have

$$\frac{1}{2}\|X\beta^\ell-y\|_{\ell_2}^2+\lambda_\ell\|\beta^\ell\|_{\ell_1} \leq \frac{1}{2}\|X\beta^\star-y\|_{\ell_2}^2+\lambda_\ell\|\beta^\star\|_{\ell_1}.$$

It yields

$$\frac{1}{2}\|Xh\|_{\ell_2}^2-\langle X^\top z,h\rangle+\lambda_\ell\|\beta^\ell\|_{\ell_1} \leq \lambda_\ell\|\beta^\star\|_{\ell_1}.$$

Let $\mathcal{S} \subseteq \{1,\dots,p\}$, we have

$$\begin{aligned}\frac{1}{2}\|Xh\|_{\ell_2}^2+\lambda_\ell\|\beta^\ell_{\mathcal{S}^c}\|_{\ell_1} &\leq \lambda_\ell\big(\|\beta^\star_{\mathcal{S}}\|_{\ell_1}-\|\beta^\ell_{\mathcal{S}}\|_{\ell_1}\big)+\lambda_\ell\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}+\langle X^\top z,h\rangle,\\ &\leq \lambda_\ell\|h_{\mathcal{S}}\|_{\ell_1}+\lambda_\ell\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}+\lambda^0\|h\|_{\ell_1},\end{aligned}$$

using (II.1). Adding $\lambda_\ell\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}$ on both sides, it holds

$$\frac{1}{2}\|Xh\|_{\ell_2}^2+(\lambda_\ell-\lambda^0)\|h_{\mathcal{S}^c}\|_{\ell_1} \leq (\lambda_\ell+\lambda^0)\|h_{\mathcal{S}}\|_{\ell_1}+2\lambda_\ell\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}.$$

Adding $(\lambda_\ell-\lambda^0)\|h_{\mathcal{S}}\|_{\ell_1}$ on both sides, we conclude the proof. □

Using (II.6) and (II.25), it follows that

$$\frac{1}{2\lambda_\ell}\Big[\frac{1}{2}\|Xh\|_{\ell_2}^2+(\lambda_\ell-\lambda^0)\|h\|_{\ell_1}\Big] \leq \Delta\sqrt{s}\,\|Xh\|_{\ell_2}+\kappa_0\|h\|_{\ell_1}+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}. \tag{II.26}$$

It yields,

$$\Big[\frac{1}{2}\Big(1-\frac{\lambda^0}{\lambda_\ell}\Big)-\kappa_0\Big]\|h\|_{\ell_1} \le \Big(-\frac{1}{4\lambda_\ell}\|Xh\|_{\ell_2}^2+\Delta\sqrt{s}\,\|Xh\|_{\ell_2}\Big)+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1},$$
$$\le \lambda_\ell\,\Delta^2\,s+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1},$$

using the fact that the polynomial $x\mapsto -(1/4\lambda_\ell)\,x^2+\Delta\sqrt{s}\,x$ is not greater than $\lambda_\ell\,\Delta^2\,s$. This concludes the proof. □

Proof of Theorem II.6 — We begin with a standard result.

Lemma II.10 — *Let $h=\beta^\ell-\beta^\star\in\mathbb{R}^p$ and $\lambda_\ell\ge\lambda^0$. Then, for all subsets $\mathcal{S}\subseteq\{1,\dots,p\}$, it holds,*

$$\frac{1}{4\lambda_d}\Big[\|Xh\|_{\ell_2}^2+(\lambda_d-\lambda^0)\|h\|_{\ell_1}\Big]\le\|h_{\mathcal{S}}\|_{\ell_1}+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}. \tag{II.27}$$

Proof. Set $h=\beta^\star-\beta^d$. Recall that $\|X^\top z\|_{\ell_\infty}\le\lambda^0$, it yields

$$\begin{aligned}\|Xh\|_{\ell_2}^2 &\le \|X^\top Xh\|_{\ell_\infty}\|h\|_{\ell_1}\\ &=\|X^\top(y-X\beta^d)+X^\top(X\beta^\star-y)\|_{\ell_\infty}\|h\|_{\ell_1}\\ &\le(\lambda_d+\lambda^0)\|h\|_{\ell_1}\,.\end{aligned}$$

Hence we get

$$\|Xh\|_{\ell_2}^2-(\lambda_d+\lambda^0)\|h_{\mathcal{S}^c}\|_{\ell_1}\le(\lambda_d+\lambda^0)\|h_{\mathcal{S}}\|_{\ell_1}\,. \tag{II.28}$$

Since $\beta^\star$ is feasible, the tube constraint (see Figure 1.7) gives $\|\beta^d\|_{\ell_1}\le\|\beta^\star\|_{\ell_1}$. Thus,

$$\begin{aligned}\|\beta^d_{\mathcal{S}^c}\|_{\ell_1}&\le\big(\|\beta^\star_{\mathcal{S}}\|_{\ell_1}-\|\beta^d_{\mathcal{S}}\|_{\ell_1}\big)+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\\ &\le\|h_{\mathcal{S}}\|_{\ell_1}+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\,.\end{aligned}$$

Since $\|h_{\mathcal{S}^c}\|_{\ell_1}\le\|\beta^d_{\mathcal{S}^c}\|_{\ell_1}+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}$, it yields

$$\|h_{\mathcal{S}^c}\|_{\ell_1}\le\|h_{\mathcal{S}}\|_{\ell_1}+2\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\,. \tag{II.29}$$

Combining (II.28) $+\,2\lambda_d\cdot$ (II.29), we get

$$\|Xh\|_{\ell_2}^2+(\lambda_d-\lambda^0)\|h_{\mathcal{S}^c}\|_{\ell_1}\le(3\lambda_d+\lambda^0)\|h_{\mathcal{S}}\|_{\ell_1}+4\lambda_d\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\,.$$

Adding $(\lambda_d-\lambda^0)\|h_{\mathcal{S}}\|_{\ell_1}$ on both sides, we conclude the proof. □

Using (II.6) and (II.27), it follows that

$$\frac{1}{4\lambda_\ell}\Big[\|Xh\|_{\ell_2}^2+(\lambda_\ell-\lambda^0)\|h\|_{\ell_1}\Big]\le\Delta\sqrt{s}\,\|Xh\|_{\ell_2}+\kappa_0\|h\|_{\ell_1}+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}. \tag{II.30}$$

It yields,

$$\Big[\frac{1}{4}\Big(1-\frac{\lambda^0}{\lambda_\ell}\Big)-\kappa_0\Big]\|h\|_{\ell_1} \le \Big(-\frac{1}{4\lambda_\ell}\|Xh\|_{\ell_2}^2+\Delta\sqrt{s}\,\|Xh\|_{\ell_2}\Big)+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1},$$
$$\le \lambda_\ell\,\Delta^2\,s+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1},$$

using the fact that the polynomial $x\mapsto -(1/4\lambda_\ell)\,x^2+\Delta\sqrt{s}\,x$ is not greater than $\lambda_\ell\,\Delta^2\,s$. This concludes the proof. □

Proof of Theorem II.5 and Theorem II.7 — Using (II.26), we know that

$$\frac{1}{2\lambda_\ell}\Big[\frac{1}{2}\|Xh\|_{\ell_2}^2+(\lambda_\ell-\lambda^0)\|h\|_{\ell_1}\Big]\le\Delta\sqrt{s}\,\|Xh\|_{\ell_2}+\kappa_0\|h\|_{\ell_1}+\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}.$$

It follows that

$$\|Xh\|_{\ell_2}^2 - 4\lambda_\ell\,\Delta\sqrt{s}\,\|Xh\|_{\ell_2} \leq 4\lambda_\ell\,\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\,.$$

This latter is of the form $x^2 - bx \leq c$ which implies that $x \leq b + c/b$. Hence,

$$\|Xh\|_{\ell_2} \leq 4\lambda_\ell\,\Delta\sqrt{s} + \frac{\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}}{\Delta\sqrt{s}}\,.$$

The same analysis holds for Theorem II.7. □

CHAPITRE III

Designs issus de graphes expanseurs déséquilibrés

In this chapter, we show that the UDP condition encompasses the renormalized adjacency matrices of the unbalanced expander graphs. A recent work [GUV09] has shown that the latter can be efficiently and deterministically constructed from Paravaresh-Vardy codes [PV05]. We give oracle inequalities in terms of estimation error and error prediction using the lasso or the Dantzig selector. In particular, we show that they are sharp, up to a logarithmic factor, in the Parvaresh-Vardy framework.

1. Preliminaries on Unbalanced Expander Graphs

We recall some basic facts about the renormalized adjacency matrices of unbalanced expander graphs. Let us denote by $G = (A, B, E)$ a bipartite graph (see Figure 3.1) where

- ✧ the set of the left vertices is denoted by A and has size p,
- ✧ the set of the right vertices is denoted by B and has size n,
- ✧ and E is the set of the edges between A and B.

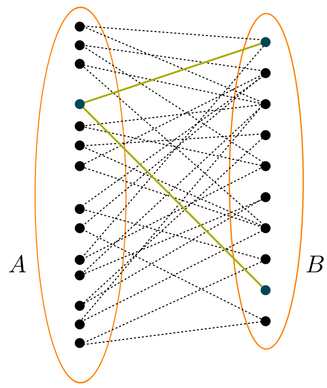


Figure 3.1. A bipartite graph G with regular left degree d. Each vertex in A has exactly d neighbors in B (here $d = 2$).

Suppose that G has regular left degree d (every vertex in A has exactly d neighbors in B), then the renormalized adjacency matrix $X \in \mathbb{R}^{n\times p}$ is

$$X_{ij} = \begin{cases} 1/d & \text{if } (j,i) \in E\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{III.1}$$

where $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, p\}$. We recall the definition of an unbalanced expander graph (see Figure 3.2).

Definition III.1 ((s, ε)-unbalanced expander) — *An (s, ε)-unbalanced expander is a bipartite simple graph $G = (A, B, E)$ with left degree d such that for any $\Omega \subset A$ with $|\Omega| \le s$,* *the set of neighbors $\Gamma(\Omega)$ of Ω has size*

$$|\Gamma(\Omega)| \geq (1-\varepsilon)\, d\, |\Omega| \; . \tag{III.2}$$

The parameter ε is called the expansion constant.

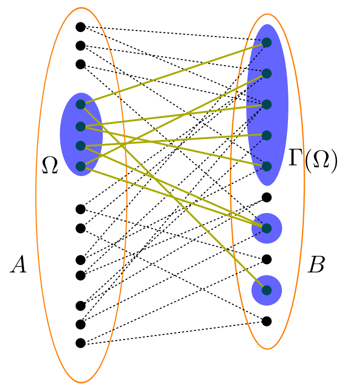


FIGURE 3.2. The expansion property of an unbalanced expander graph: any sufficiently small subset I on the left has a neighborhood J of size at least $(1-\varepsilon)\, d\, |I|$.

Subsequently we shall work with $\varepsilon = 1/12$. Notice that ε is fixed and does not depend on others parameters and that it does not go to zero as p goes to the infinity.

1.1. The uncertainty principle of the unbalanced expander graphs. In 2008, the paper [BGI+08] showed that the adjacency matrix of an expander graph satisfies a very similar property to the RIP property (see Section 2.4, page 9), called the restricted isometry property in the ℓ_1-norm. Indeed, they showed the fundamental theorem:

Theorem III.1 ([BGI+08], RIP_1 for Unbalanced Expander Graphs) — *Let $X \in \mathbb{R}^{n\times p}$ be the renormalized adjacency matrix of an (s,ε)-unbalanced expander. Then X satisfies the following* RIP_1 *property:*

$$\forall \gamma \in \mathbb{R}^p, \;\; (1-2\varepsilon)\|\gamma_S\|_{\ell_1} \leq \|X\gamma_S\|_{\ell_1} \leq \|\gamma_S\|_{\ell_1} \, ,$$

where S is any subset of $\{1,\dots,p\}$ of size less than s, and γ_S the vector with coefficients equal to the coefficients of γ in S and zero outside.

They showed that exact recovery using basis pursuit (with unbalanced expander graph designs) is still possible. Moreover, they proved ([BGI+08], Lemma 16 and Theorem 17) a useful uncertainty principle connecting the mass on a small subset S, namely $\|\gamma_S\|_{\ell_1}$, to the whole mass $\|\gamma\|_{\ell_1}$.

Lemma III.2 ([BGI+08], Uncertainty Principle) — *Let $X \in \mathbb{R}^{n\times p}$ be the renormalized adjacency matrix of an $(2s,\varepsilon)$-unbalanced expander with $\varepsilon < 1/4$. Then X satisfies the following uncertainty principle:*

$$\forall \gamma \in \mathbb{R}^p, \; \forall S \subseteq \{1,\dots,p\} \text{ s.t. } |S| \leq s, \;\; (1-4\varepsilon)\|\gamma_S\|_{\ell_1} \leq \|X\gamma\|_{\ell_1} + 2\varepsilon\|\gamma_{S^c}\|_{\ell_1} \, , \tag{III.3}$$

where γ_S denotes the vector of which i-th entry is equal to γ_i if $i \in S$ and 0 otherwise.

Observe that this property is very similar to the UDP property.

Lemma III.3 — *Let $X \in \mathbb{R}^{n\times p}$ be the renormalized adjacency matrix of an $(2s,\varepsilon)$-unbalanced expander with $\varepsilon < 1/10$. Then X satisfies* $\mathrm{UDP}(s,\kappa_0,\Delta)$ *with*

$$\kappa_0 = \frac{2\varepsilon}{1-2\varepsilon} \quad \text{and} \quad \Delta = \frac{\sqrt{n}}{(1-2\varepsilon)\sqrt{s}}\,.$$

Observe that κ_0 is less than $1/4$ for $\varepsilon < 1/10$.

Proof. Assume that (III.3) holds then

$$\|\gamma_S\|_{\ell_1} \leq \frac{1}{1-2\varepsilon}\|X\gamma\|_{\ell_1} + \frac{2\varepsilon}{1-2\varepsilon}\|\gamma\|_{\ell_1} \leq \frac{\sqrt{n}}{1-2\varepsilon}\|X\gamma\|_{\ell_2} + \frac{2\varepsilon}{1-2\varepsilon}\|\gamma\|_{\ell_1}\,.$$

This concludes the proof. □

In particular, if $\varepsilon = 1/12$ then X satisfies $\mathrm{UDP}(s,1/5,6\sqrt{n}/5\sqrt{s})$, see Table 1.

$$\boxed{\Big(2s,\frac{1}{12}\Big)\text{-unbalanced expander} \Longrightarrow \mathrm{UDP}\Big(s,\frac{1}{5},\frac{6\sqrt{n}}{5\sqrt{s}}\Big)}$$

Table 1. The renormalized adjacency matrix of a $(2s,1/12)$-unbalanced expander graph satisfies $\mathrm{UDP}(s,1/5,6\sqrt{n}/5\sqrt{s})$.

1.2. The UDP condition and the Juditsky-Nemirovski condition. In parallel to our work, A. Juditsky and A. Nemirovski [JN11] gave an remarkable efficiently verifiable condition of performance of the lasso and the Dantzig selector. Although the matrices constructed from the expander graphs are not specifically studied in [JN11], they study uncertainty conditions similar to the ones stated in equation (III.6). An attentive reading of their article shows that the $\mathbf{H}_{s,1}(1/5)$ condition is related to UDP condition (III.6). Indeed, the $\mathbf{H}_{s,1}(1/5)$ condition (see 5.3 in [JN11]) for the lasso and the Dantzig selector, is given by:

$$\forall\gamma\in\mathbb{R}^p,\ \forall S\subseteq\{1,\dots,p\}\ \text{s.t.}\ |S|\leq s\,,\quad \|\gamma_S\|_{\ell_1}\leq \hat\lambda\, s\,\|X\gamma\|_{\ell_2}+\frac{1}{3}\|\gamma\|_{\ell_1}\,,$$

where

$$\hat\lambda = \frac{6\sqrt{n}}{5s}\,. \tag{III.4}$$

Their result (Proposition 9, [JN11]) is similar to (III.9) and (III.14) in terms of regular consistency (see the discussion in Section 3.3). However, let us emphasize that the results in [JN11] concerns regular consistency for ℓ_1-recovery. In particular, there is no result in error prediction.

✧ In addition, the authors established limits of performance on their conditions: the condition $\mathbf{H}_{s,\infty}(1/5)$ (that implies $\mathbf{H}_{s,1}(1/5)$) is feasible only in a severe restricted range of the sparsity parameter s. The reader may find a complete discussion on this subject in [JN11].

1.3. Deterministic design. A long standing issue is to give efficient and deterministic construction of unbalanced expander graphs, see [HX07, CHJX09] for instance. Using Parvaresh-Vardy codes [PV05], V. Guruswami, C. Umans, and S. Vadhan have recently proved the following theorem.

Theorem III.4 ([GUV09], Explicit construction) — *There exists an universal constant $\theta_0 > 0$ such that the following holds. For all $\alpha > 0$ and for all $p, s, \varepsilon > 0$, there exists an (s, ε)-unbalanced expander graph $G = (A, B, E)$ with $|A| = p$, left degree*

$$d \leq \big((\theta_0 \log p \log s)/\varepsilon\big)^{1+\frac{1}{\alpha}},$$

and right side vertices (of size $n = |B|$) such that

$$n \leq s^{1+\alpha}\big((\theta_0 \log p \log s)/\varepsilon\big)^{2+\frac{2}{\alpha}}. \tag{III.5}$$

Notice that the size n may depend on p and others parameters of the graph. As mentioned in the introduction, all the results in this chapter hold for the following deterministic design construction:

(1) Choose p the size of the target, and s the sparsity level,
(2) Set $\varepsilon = 1/12$ the expansion constant, and $\alpha > 0$ a tunning parameter,
(3) Construct an (s, ε)-unbalanced expander graph G from Paravaresh-Vardy codes (with parameter α).
(4) Set $X \in \mathbb{R}^{n\times p}$ the renormalized adjacency matrix of the graph G. Notice that the number of observations n satisfies (III.5).

In this framework, the number of measurements n depends on the number of predictors p and the sparsity constant s.

Remark — Yet it does not match the random ones, the explicit constructions are not too far in terms of the bounds on n.

Proposition III.5 ([HX07], Probabilistic construction) — *Consider $\varepsilon > 0$ and $p/2 \geq s$. Then, with a positive probability, there exists an (s, ε)-unbalanced expander graph $G = (A, B, E)$ with $|A| = p$, left degree*

$$d = \underset{p\to+\infty}{\mathcal{O}}\big(\log(p/s)\big),$$

and number of right side vertices (namely $n = |B|$),

$$n = \underset{p\to+\infty}{\mathcal{O}}\big(s\log(p/s)\big),$$

where the $\mathcal{O}(.)$ notation does not depend on s but on ε.

2. Oracle inequalities for the lasso and the Dantzig selector

Consider the Gaussian linear model defined by (I.2). We recall that the definition of the lasso (resp. the Dantzig selector) can be found in (I.22) (resp. (I.23)). Consider a design matrix $X \in \mathbb{R}^{n\times p}$ that satisfies two conditions, namely:

- ✦ The ℓ_1-normalization condition: *All the columns of the design matrix $X \in \mathbb{R}^{n\times p}$ have ℓ_1-norm equal to 1.*
- ✦ The UDP$(s, 1/5, 6\sqrt{n}/5\sqrt{s})$ condition:

$$\forall \gamma \in \mathbb{R}^p,\ \forall S \subseteq \{1, \ldots, p\} \text{ s.t. } |S| \leq s,\quad \|\gamma_S\|_{\ell_1} \leq \frac{6\sqrt{n}}{5}\|X\gamma\|_{\ell_2} + \frac{1}{5}\|\gamma\|_{\ell_1} \tag{III.6}$$

✧ Notice that the renormalized adjacency matrix of an unbalanced expander graphs with expansion constant ε not greater than $1/12$ satisfies them (see Lemma III.2 and Table 1). Unless otherwise specified, we assume that the design matrix $X \in \mathbb{R}^{n\times p}$ satisfies these two conditions.

2.1. Error prediction and estimation error for the lasso. We have the following oracle inequalities for the lasso, derived from Theorem II.4 and Theorem II.5 page 25-26.

Theorem III.6 ([dC10]) **—** *Let $X \in \mathbb{R}^{n\times p}$ be such that*

✦ *It satisfies the* $\mathrm{UDP}(s, 1/5, 6\sqrt{n}/5\sqrt{s})$ *condition,*

✦ *it satisfies the Bound on the noise condition:*

$$\mathbb{P}(\|X^\top z\|_{\ell_\infty} \leq \lambda^0) \geq 1 - \eta_n, \tag{III.7}$$

where η_n is some known function that depends only on n and $\lambda^0 = 2\,\sigma_n\,\sqrt{\log n}$.

Then for any

$$\lambda_\ell > 5\lambda^0/3\,, \tag{III.8}$$

it holds

$$\|\beta^\ell - \beta^\star\|_{\ell_1} \leq \frac{2}{(1-\frac{\lambda^0}{\lambda_\ell}) - \frac{2}{5}} \cdot \Big[\frac{36}{25}\cdot\lambda_\ell\cdot n + \min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s.}} \|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big]. \tag{III.9}$$

with probability at least $1-\eta_n$. Similarly, it holds

$$\|X\beta^\ell - X\beta^\star\|_{\ell_2} \leq \frac{24}{5}\cdot\lambda_\ell\cdot\sqrt{n} + \frac{5}{6\sqrt{n}}\cdot \min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\ |\mathcal{S}|=s.}} \|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}. \tag{III.10}$$

with probability at least $1-\eta_n$.

✧ In Section 4, we show that the ℓ_1-normalization condition implies the Bound on the noise condition.

✧ These oracle inequalities give the error prediction and the estimation error in terms of the s-best term approximation, namely $\|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}$.

✧ Assume that $\beta^\star$ is s-sparse and that $\lambda_\ell = 2\lambda^0$. Then (III.10) gives

$$\|X\beta^\ell - X\beta^\star\|_{\ell_2} \leq 20\cdot\sigma_n\cdot\sqrt{n\log n}\,. \tag{III.11}$$

Similarly, the regular consistency can be upper bounded by

$$\|\beta^\ell - \beta^\star\|_{\ell_1} \leq 58\cdot\sigma_n\cdot n\sqrt{\log n}\,. \tag{III.12}$$

In Section 3, we show that this inequality is sharp, up to a logarithmic factor, in the Paravaresh-Vardy code framework.

2.2. Error prediction and estimation error for the Dantzig Selector. Besides we have the following oracle inequalities for the Dantzig selector.

Theorem III.7 ([dC10]) **—** *Let $X \in \mathbb{R}^{n\times p}$ be such that*

✦ *It satisfies the* $\mathrm{UDP}(s, 1/5, 6\sqrt{n}/5\sqrt{s})$ *condition,*

✦ *it satisfies the Bound on the noise condition:*

$$\mathbb{P}(\|X^\top z\|_{\ell_\infty} \leq \lambda^0) \geq 1 - \eta_n, \qquad \text{(III.7)}$$

where η_n is some known function that depends only on n and $\lambda^0 = 2\,\sigma_n\,\sqrt{\log n}$.

Then for any

$$\lambda_\ell > 5\lambda^0\,, \tag{III.13}$$

it holds

$$\|\beta^d - \beta^\star\|_{\ell_1} \leq \frac{4}{(1-\frac{\lambda^0}{\lambda_\ell}) - \frac{4}{5}} \cdot \Big[\frac{36}{25}\cdot\lambda_\ell\cdot n + \min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\|\mathcal{S}|=s.}} \|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}\Big]. \tag{III.14}$$

with probability at least $1-\eta_n$*. Similarly, it holds*

$$\|X\beta^d - X\beta^\star\|_{\ell_2} \leq \frac{24}{5}\cdot\lambda_\ell\cdot\sqrt{n} + \frac{5}{6\sqrt{n}}\cdot \min_{\substack{\mathcal{S}\subseteq\{1,\ldots,p\},\\|\mathcal{S}|=s.}} \|\beta^\star_{\mathcal{S}^c}\|_{\ell_1}. \tag{III.15}$$

with probability at least $1-\eta_n$*.*

Observe that, in the sparse case, the Dantzig selector satisfies (III.11) and, up to a multiplicative constant, (III.12).

3. Results in the Parvaresh-Vardy code framework

In this section we study the bounds appearing in the previous oracle inequalities. In particular, we recall that there exists a construction of design $X\in\mathbb{R}^{n\times p}$ derived from Parvaresh-Vardy codes such that the following holds [GUV09]: There exists an universal constant $\theta>0$ such that, for all $\alpha>0$, $p\geq s>0$, there exists an explicit renormalized adjacency matrix $X\in\mathbb{R}^{n\times p}$ of unbalanced expander graph (with an expansion constant $\varepsilon=1/12$) such that,

(i) $n\leq s^{1+\alpha}(\theta\log p\log s)^{2+\frac{2}{\alpha}}$,

(ii) the left degree d of the graph satisfies $d\leq(\theta\log p\log s)^{1+\frac{1}{\alpha}}$,

(iii) the matrix X satisfies the ℓ_1-normalization condition,

(iv) the columns $X_i\in\mathbb{R}^n$ of the matrix $X\in\mathbb{R}^{n\times p}$ are such that $\|X_i\|_{\ell_2}=1/\sqrt{d}$,

(v) the matrix X satisfies the UDP$(s,1/5,6\sqrt{n}/5\sqrt{s})$ condition (III.6).

Notice that :

- ✧ Condition (i) and (ii) are derived from Theorem III.4 (the constant θ is exactly $\theta_0/\varepsilon=8\theta_0$),
- ✧ Condition (iii) and (iv) are derived from the definition of a renormalized adjacency matrix (III.1),
- ✧ Condition (v) is a consequence of Lemma III.2 (where the expansion constant is such that $\varepsilon=1/12$).

In this section, we assume that the design $X\in\mathbb{R}^{n\times p}$ satisfies the five above conditions. As a matter of fact, Inequality (III.11) shows that we can estimate $X\beta^\star\in\mathbb{R}^n$ with nearly the same precision as if one knew in advance the support of $\beta^\star\in\mathbb{R}^p$. As before, consider the ordinary least square estimator:

$$\beta^{ideal} = \arg\min_{\substack{\beta\in\mathbb{R}^p\\ \mathrm{supp}(\beta)=\mathcal{S}}} \|y-X\beta\|_{\ell_2}\,,$$

where $\mathcal{S}_\star$ denotes the support of the target $\beta^\star\in\mathbb{R}^p$. Observe that this estimator uses a prior knowledge on the support of $\beta^\star$. For this reason, we can say that this estimator is *ideal*. A simple calculation gives

$$\frac{1}{n}\,\mathbb{E}\|X\beta^{ideal} - X\beta^\star\|^2_{\ell_2} = \sigma_n^2\,\frac{s}{n}\,.$$

Observe that a very similar analysis has been given in Section 3.4, page 17. Using (i) we deduce that (III.11) is optimal up to an explicit multiplicative factor $\rho(s,p)$. Namely, it holds, with high probability,

$$\frac{1}{n}\|X\beta^\star - X\beta^\ell\|_{\ell_2}^2 \leq C\cdot\rho(s,p)\cdot\frac{1}{n}\,\mathbb{E}\|X\beta^{ideal} - X\beta^\star\|_{\ell_2}^2\,. \tag{III.16}$$

where $\rho(s,p) = \big((1+\alpha)\log s + (2+2/\alpha)\log(\theta\,\log p\log s)\big)\cdot s^\alpha(\theta\,\log p\log s)^{2+\frac{2}{\alpha}}$, and $C>0$ is a numerical constant. This inequality shows that prediction using Parvaresh-Vardy code design is almost optimal. Indeed, the prediction error is, up to the factor $\rho(s,p)$, as good as the error prediction one would have get knowing the support of the target. Furthermore, notice that the same comment holds for the Dantzig selector (see Section 2.2). As a matter of fact, all the comments of Section 3 extend to the Dantzig selector. Notice that

$$\forall i\in\{1,\ldots,p\},\quad \|X_i\|_{\ell_2}^2 = \|X_1\|_{\ell_2}^2 \geq (\theta\log p\log s)^{-1-\frac{1}{\alpha}}\,.$$

In order to compare the results of this chapter to the standard results given by the Restricted Eigenvalue assumption [**BRT09**] and the coherence property [**DET06**], we derive from (III.16) the following inequality, with high probability:

$$\frac{1}{n}\|X\beta^\star - X\beta^\ell\|_{\ell_2}^2 \leq C\,.\,\tau(s,p)\,.\,\sigma_n^2\,\|X_1\|_2^2\,\frac{s\log p}{n}\,. \tag{III.17}$$

where $\tau(s,p) = s^\alpha(\theta\,\log p\log s)^{3+\frac{3}{\alpha}}\,.\,\big(\log\big(s^{1+\alpha}(\theta\,\log p\log s)^{2+\frac{2}{\alpha}}\big)/\log p\big)$, and the numerical constant $C>0$ is the same as in the previous inequality.

3.1. Comparison with the coherence property approach. In 2007, E. J. Candès and Y. Plan obtained a remarkable estimate in error prediction for the lasso. They used a so-called coherence property following the work of D.L. Donoho *et al.* [**DET06**]. They showed (Theorem 1.2 in [**CP09**]) that, with high probability, for every design matrix satisfying the coherence property, it holds

$$\frac{1}{n}\|X\beta^\star - X\beta^\ell\|_{\ell_2}^2 \leq C'\,.\,\sigma_n^2\,\|X_1\|_2^2\,\frac{s\log p}{n}\,, \tag{III.18}$$

where $C'>0$ is some positive numerical constant. Note that the upper bounds (III.17) and (III.18) are similar up to the factor $\tau(s,p)$. The coherence is the maximum correlation between pairs of predictors. This property is fundamental and allows to deal with random design matrices. We do not use this property here, though we get the same accuracy (up to the factor $\tau(s,p)$) and we extend their error prediction result to deterministic design matrices.

3.2. Comparison with the Restricted Eigenvalue approach. In the same way, P.J. Bickel, Y. Ritov, and A.B. Tsybakov [**BRT09**] established that, with high probability,

$$\frac{1}{n}\|X\beta^\star - X\beta^\ell\|_{\ell_2}^2 \leq C''\,.\,\sigma_n^2\,\|X_1\|_2^2\,\frac{s\log p}{n}\,,$$

where $C''>0$ is some positive constant depending on the $(s,3)$-restricted ℓ_2-eigenvalue, $\kappa(s,3)$. Again, it is difficult to estimate $\kappa(s,3)$ for the adjacency matrix of an unbalanced expander graph. Observe that, up to the factor $\tau(s,p)$, we get same accuracy.

3.3. Comparison with the Juditsky-Nemirovski approach. As mentioned in Section 1.2, the $\mathbf{H}_{s,1}(1/3)$ condition is devoted to regular consistency for the lasso and the Dantzig selector. In particular, the results in [JN11] should be compared to the result (III.12) in the previous corollary:

$$\|\beta^\ell - \beta^\star\|_{\ell_1} \leq 58 \cdot \sigma_n \cdot s^{1+\alpha}(\theta \log p \log s)^{\frac{3}{2}+\frac{3}{2\alpha}} \sqrt{\log\left(s^{1+\alpha}(\theta \log p \log s)^{2+\frac{2}{\alpha}}\right)} \cdot \|X_1\|_{\ell_2}, \tag{III.19}$$

using (i). Following Proposition 8 in [JN11] (with $\beta = \|X_1\|_{\ell_2}$, $\kappa = 1/5$, $p = 1$, $\epsilon = \eta_n$, and $\hat{\lambda}$ as in (III.4)), A. Juditsky and A. Nemirovski show that

$$\|\beta^\ell - \beta^\star\|_{\ell_1} \leq 256\sqrt{2} \cdot \sigma_n \cdot s^{1+\alpha}(\theta \log p \log s)^{2+\frac{2}{\alpha}} \cdot \sqrt{\log(p/\eta_n)} \cdot \|X_1\|_{\ell_2}, \tag{III.20}$$

where η_n is given by the Bound on the noise condition (III.7). Up to a logarithmic factor, the result (III.20) is of the same order than the result (III.19).

4. Bound on the graph noise

In this section, we give an upper bound on the noise amplification $\|X^\top z\|_{\ell_\infty}$. In particular, we assume the ℓ_1-normalization condition and we show that the Bound on the noise condition (III.7) holds.

Lemma III.8 (Non-Amplification) — *It holds* $\forall z \in \mathbb{R}^n$, $\|X^\top z\|_{\ell_\infty} \leq \|z\|_{\ell_\infty}$.

Proof. Let $\gamma \in \mathbb{R}^p$ such that $\|\gamma\|_{\ell_1} = 1$. Since the design matrix satisfies the ℓ_1-normalization condition, the triangular inequality gives that $\|X\gamma\|_{\ell_1} \leq \|\gamma\|_{\ell_1}$. Furthermore,

$$\|X^\top z\|_{\ell_\infty} \leq \max_{\|\gamma\|_{\ell_1}\leq 1} \langle X^\top z, \gamma\rangle = \max_{\|\gamma\|_{\ell_1}\leq 1} \langle z, X\gamma\rangle \leq \max_{\|\gamma\|_{\ell_1}\leq 1} \{\|z\|_{\ell_\infty}\|X\gamma\|_{\ell_1}\} \leq \|z\|_{\ell_\infty},$$

where $\langle\, ,\, \rangle$ is the standard Euclidean product. □

In order to upper bound $\|X^\top z\|_{\ell_\infty}$, it is enough to estimate $\|z\|_{\ell_\infty}$. This comment allows us to reduce the dimension of the ambient space from p to n. In comparizon, Lemma II.1 (page 21) contains a $\log(p)$ factor that does not exist in the next lemma.

Lemma III.9 (Bound on the noise) — *Suppose that $z = (z_i)_{i=1}^n$ is a centered Gaussian noise such that the z_i's could be correlated, and for all $i \in \{1, \dots, n\}$, we have $z_i \sim \mathcal{N}(0, \sigma^2)$.* ***Then**, for $\lambda^0 = 2\sigma\sqrt{\log n}$,*

$$\mathbb{P}(\|X^\top z\|_{\ell_\infty} \leq \lambda^0) \geq 1 - \frac{1}{\sqrt{2\pi}\, n\sqrt{\log n}}.$$

Proof. Denote $(z_i)_{i=1\dots n}$ the coefficients of z. Lemma III.8 gives

$$\mathbb{P}(\|X^\top z\|_{\ell_\infty} \leq \lambda^0) \geq \mathbb{P}\left(\|z\|_{\ell_\infty} \leq \lambda^0\right). \tag{III.21}$$

Using Šidák's inequality in (III.21), it holds [Š68]:

$$\mathbb{P}\left(\|z\|_{\ell_\infty} \leq \lambda^0\right) \geq \mathbb{P}\left(\|\tilde{z}\|_{\ell_\infty} \leq \lambda^0\right) = \prod_{i=1}^n \mathbb{P}\left(|\tilde{z}_i| \leq \lambda^0\right),$$

where the $\tilde{z}_i$'s are independent and have the same law as the z_i's. Denote Φ and φ respectively the *cumulative distribution function* and the probability density function of the standard normal. Set $\delta = 2\sqrt{\log n}$. It holds

$$\prod_{i=1}^{n} \mathbb{P}\left(|\tilde{z}_i| \leq \lambda^0\right) = \mathbb{P}\left(|z_1| \leq \lambda^0\right)^n = (2\Phi(\delta) - 1)^n > \left(1 - 2\varphi(\delta)/\delta\right)^n,$$

using an integration by parts to get $1 - \Phi(\delta) < \varphi(\delta)/\delta$. It yields that

$$\mathbb{P}\left(\|X^\top z\|_{\ell_\infty} \leq \lambda^0\right) \geq \left(1 - 2\varphi(\delta)/\delta\right)^n \geq 1 - 2n\frac{\varphi(\delta)}{\delta} = 1 - \frac{1}{\sqrt{2\pi}\, n\sqrt{\log n}}.$$

This concludes the proof. □

This upper bound is valuable to give oracle inequalities, as seen in the previous sections. For readability sake, denote

$$\eta_n = \frac{1}{\sqrt{2\pi}\, n\sqrt{\log n}}.$$

All the probabilities appearing in our theorems are of the form $1 - \eta_n$. Since n denote the number of observations, η_n is very small (less than $1/1000$ for most common problems). Furthermore, by repeating the same argument as in Lemma III.9, we have the next proposition.

Proposition III.10 — *Suppose that $z = (z_i)_{i=1}^n$ is a centered Gaussian noise with variance σ^2 such that the z_i's are $\mathcal{N}(0, \sigma^2)$-distributed and could be correlated.*
***Then**, for $t \geq 1$ and*

$$\lambda^0(t) = (1+t)\,\sigma\sqrt{\log n},$$

$$\mathbb{P}\left(\|X^\top z\|_{\ell_\infty} \leq \lambda^0(t)\right) \geq 1 - \frac{\sqrt{2}}{(1+t)\sqrt{\pi \log n}\, n^{\frac{(1+t)^2}{2}-1}}. \tag{III.22}$$

By replacing λ^0 by $\lambda^0(t)$ in the statements of our theorems, it is possible to replace all the probabilities of the form $1 - \eta_n$ by probabilities of the form (III.22). Observe that these probabilities can be as small as desired.

CHAPITRE IV

Relaxation convexe sur l'espace des mesures signées

In this chapter, we are interested in the measure framework. We show that one can reconstruct a measure with finite support from few non-adaptive linear measurements. Surprisingly our method, called *support pursuit*, can be seen as an extension of basis pursuit (see (I.11) for a definition). More precisely, consider a signed discrete measure σ on a set I. Unless otherwise specified, assume that $I := [-1, 1]$. Notice that all our results easily extend to any real bounded set. Consider the Jordan decomposition of σ:

$$\sigma = \sigma^+ - \sigma^-,$$

where σ^+ and σ^- are nonnegative measures such that there exists two disjoints measurable sets $\mathcal{S}^+$ and $\mathcal{S}^-$ such that for all measurable set $E \subseteq \mathbb{R}$, $\sigma^+(E) = \sigma(E \cap \mathcal{S}^+)$ and $\sigma^- = \sigma(E \cap \mathcal{S}^-)$. These measures are essentially unique in the sense that the sets $\mathcal{S}^+$ and $\mathcal{S}^-$ are uniquely defined up to a σ-null set. Define the *Jordan support* of the measure σ as the pair $\mathcal{J} := (\mathcal{S}^+, \mathcal{S}^-)$. Assume further that $\mathcal{S} := \mathcal{S}^+ \cup \mathcal{S}^-$ is finite and has cardinality equal to s. Moreover suppose that $\mathcal{J}$ belongs to a family Υ of pairs of subsets of I (see Definition IV.1 for more details). We call Υ a *Jordan support family*. The measure σ can be written as

$$\sigma = \sum_{i=1}^{s} \sigma_i \, \delta_{x_i} \,,$$

where $\mathcal{S} = \{x_1, \ldots, x_s\}$, $\sigma_1, \ldots, \sigma_s$ are nonzero real numbers and δ_x denotes the Dirac measure at point x. Let $\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ be any family of continuous functions on $\bar{I}$, where the set $\bar{I}$ denotes the closure of I. Let μ be a signed measure on I, the k-th generalized moment of μ is defined by

$$c_k(\mu) = \int_I u_k \, \mathrm{d}\mu \,, \tag{IV.1}$$

for all the indices $k = 0, 1, \ldots, n$. We are concerned by the reconstruction of the target measure σ from the observation of $\mathcal{K}_n(\sigma) := (c_0(\sigma), \ldots, c_n(\sigma))$, i.e. its $(n+1)$ first generalized moments. We assume that both the support $\mathcal{S}$ and the weights σ_i of the target measure σ are unknown. We investigate the reconstruction of σ from the observation of $\mathcal{K}_n$. More precisely, does an algorithm fitting $\mathcal{K}_n(\sigma)$ among all the signed measures of I recover the measure σ? Remark that a finite number of assigned standard moments does not define a unique signed measure. In fact one can check that for all signed measures and for all integers m there exists a distinct signed measure with the same m first standard moments. It seems there is no hope in recovering discrete measures from a finite number of its generalized moments. Surprisingly, we show that every *extremal Jordan type* measure σ (see Definition IV.1 and the examples that follows) is the unique solution of a total variation minimizing algorithm, the support pursuit.

1. The support pursuit

Denote by $\mathcal{M}$ the set of the finite signed measures on I and by $\|\,.\,\|_{TV}$ the total variation norm. We recall that, for all $\mu \in \mathcal{M}$,

$$\|\mu\|_{TV} = \sup_{\Pi} \sum_{E\in\Pi} |\mu(E)| \ ,$$

where the supremum is taken over all partition Π of I into a finite number of disjoint measurable subsets. Given the observation $\mathcal{K}_n(\sigma) = (c_0(\sigma), \dots, c_n(\sigma))$, the support pursuit is

$$\sigma^\star \in \operatorname*{Arg\,min}_{\mu\in\mathcal{M}} \|\mu\|_{TV} \quad \text{s.t. } \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma)\,. \tag{IV.2}$$

On one hand, basis pursuit minimizes the ℓ_1-norm subject to linear constraints. On the other hand, support pursuit naturally substitutes the TV-norm (the total variation norm) for the ℓ_1-norm. Let us emphasize that support pursuit looks for a minimizer among all the signed measures on I. Nevertheless, the target measure σ is assumed to be of extremal Jordan type.

Definition IV.1 (Extrema Jordan type measure) — *We say that a signed measure μ is of extremal Jordan type (with respect to a family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$) if and only if its Jordan decomposition $\mu = \mu^+ - \mu^-$ satisfies*

$$\operatorname{Supp}(\mu^+) \subset E_P^+ \quad \text{and} \quad \operatorname{Supp}(\mu^-) \subset E_P^-,$$

where $\operatorname{Supp}(\nu)$ *is defined as the support of the measure ν, and*
- ✦ *P denotes any linear combination of elements of $\mathcal{F}$,*
- ✦ *P is not constant and $\|P\|_\infty \le 1$,*
- ✦ *E_P^+ (resp. E_P^-) is the set of all points x_i such that $P(x_i) = 1$ (resp. $P(x_i) = -1$).*

In the following, we give some examples of extremal Jordan type measures with respect to the family

$$\mathcal{F}_p^n = \{1, x, x^2, \dots, x^n\}\,.$$

These measures can be seen as "interesting" target measures for support pursuit given the observation of the $n+1$ first standard moments. For sake of readability, let n be an even integer $n = 2m$. We present three important examples.

- ✦ **Nonnegative measures:** The nonnegative measures, of which support has size s not greater than $n/2$, are extremal Jordan type measures. Indeed, let σ be a nonnegative measure and $\mathcal{S} = \{x_1, \dots, x_s\}$ be its support. Set

  $$P(x) = 1 - c\prod_{i=1}^{s}(x - x_i)^2\,.$$

  Then, for a sufficiently small value of the parameter c, the polynomial P has supremum norm not greater than 1. The existence of such polynomial shows that the measure σ is an extremal Jordan type measure. In Section 3 we extend this notion to any *homogeneous M-system*.

- ✦ **Chebyshev measures:** The k-th Chebyshev polynomial of the first order is defined by

  $$\forall x \in [-1, 1], \quad T_k(x) = \cos(k \arccos(x))\,. \tag{IV.3}$$

  It is well known that it has supremum norm not greater than 1, and that

  - ✧ $E_{T_k}^+ = \big\{ \cos(2l\pi/k)\,;\ l = 0, \dots, \lfloor \frac{k}{2} \rfloor \big\}$,

✧ $E^-_{T_k} = \{ \cos((2l+1)\pi/k)\,;\ l = 0, \ldots, \lfloor \frac{k}{2} \rfloor \}$,

Then, any measure σ such that

$$\mathrm{Supp}(\sigma^+) \subset E^+_{T_k} \quad \text{and} \quad \mathrm{Supp}(\sigma^-) \subset E^-_{T_k},$$

for some $0 < k \leq n$, is an extremal Jordan type measure. Further examples are presented in Section 4.

✦ **Δ-spaced out type measures:** Let Δ be a positive real and S_Δ be the set of all pairs (S^+, S^-) of subsets of $[-1,1]$ such that

$$\forall x, y \in S^+ \cup S^-,\ x \neq y, \quad |x - y| \geq \Delta.$$

In Lemma IV.10, we prove that for all $(S^+, S^-) \in S_\Delta$ there exists a polynomial $P_{(S^+,S^-)}$ such that

✧ $P_{(S^+,S^-)}$ has degree n not greater than a bound depending only on Δ,

✧ $P_{(S^+,S^-)}$ is equal to 1 on the set S^+,

✧ $P_{(S^+,S^-)}$ is equal to -1 on the set S^-,

✧ and $\| P_{(S^+,S^-)} \|_\infty \leq 1$.

This shows that any measure σ with Jordan support included in S_Δ is an extremal Jordan type measure.

In this chapter, we give exact reconstruction results for these three kinds of extremal Jordan type measures. As a matter of fact, our results extend to others family $\mathcal{F}$. Roughly, they can be stated as follows:

✦ **Nonnegative measures:** Assume that $\mathcal{F}$ is a homogeneous M-system (see 3.1). Theorem IV.2 shows that any nonnegative measure σ is the unique solution of support pursuit given the observation $\mathcal{K}_n(\sigma)$ where n is not less than twice the size of the support of σ.

✦ **Generalized Chebyshev measures:** Assume that the $\mathcal{F}$ is a M-system (see definition 3.1). Proposition IV.8 shows the following result: Let σ be a signed measure having Jordan support included in $(E^+_{\mathfrak{T}_k}, E^-_{\mathfrak{T}_k})$, for some $1 \leq k \leq n$, where $\mathfrak{T}_k$ denotes the k-th generalized Chebyshev polynomial (see 4.3). Then σ is the unique solution to support pursuit (IV.2) given $\mathcal{K}_n(\sigma)$, i.e. its $(n+1)$ first generalized moments.

✦ **Δ-interpolation:** Considering the standard family $\mathcal{F}^n_p = \{1, x, x^2, \ldots, x^n\}$, Proposition IV.11 shows that support pursuit exactly recovers any Δ-spaced out type measure σ from the observation $\mathcal{K}_n(\sigma)$ where n is greater than a bound depending only on Δ.

These results are closely related to standard results of basis pursuit [Dono06].

2. The generalized dual polynomials

In this section we introduce the generalized dual polynomial. In particular we give a sufficient condition that guarantees the exact reconstruction of the measure σ. As a matter of fact, this condition relies on an interpolation problem.

2.1. An interpolation problem. An insight into exact reconstruction is given by Lemma IV.1. It shows that the existence of a generalized dual polynomial is a sufficient condition to the exact reconstruction of a signed measure with finite support. The following results holds for any family $\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ of continuous function on $\bar{I}$. We recall that throughout this thesis, $\mathrm{sgn}(x)$ denotes the sign of the real x.

Lemma IV.1 (The generalized dual polynomials) — *Let n be a nonzero integer. Let $\mathcal{S} = \{x_1, \dots, x_s\} \subset I$ be a subset of size s and $(\varepsilon_1, \dots, \varepsilon_s) \in \{\pm 1\}^s$ be a sign sequence. If there exists a linear combination $P = \sum_{k=0}^{n} a_k u_k$ such that*

(i) the generalized Vandermonde system

$$\begin{pmatrix} u_0(x_1) & u_0(x_2) & \dots & u_0(x_s) \\ u_1(x_1) & u_1(x_2) & \dots & u_1(x_s) \\ \vdots & \vdots & & \vdots \\ u_n(x_1) & u_n(x_2) & \dots & u_n(x_s) \end{pmatrix}$$

has full column rank.

(ii) $\forall\, i = 1, \dots, s, \quad P(x_i) = \varepsilon_i$,

(iii) $\forall x \in [-1, 1] \setminus \mathcal{S}, \quad |P(x)| < 1$,

Then every measure $\sigma = \sum_{i=1}^{s} \sigma_i \, \delta_{x_i}$, such that $\mathrm{sgn}(\sigma_i) = \varepsilon_i$, is the unique solution to support pursuit given the observation $\mathcal{K}_n(\sigma)$.

Proof. See page 62. □

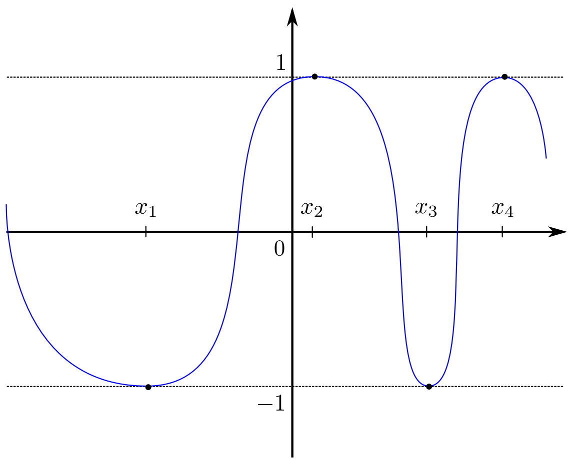


Figure 4.1. An example of dual polynomial with $s = 4$.

The linear combination P considered in the Lemma IV.1 is called a *generalized dual polynomial*. This naming is inherited from the original article [CRT06a] of E.J. Candès, T. Tao and J. Romberg, and from the dual certificate named by E.J. Candès and Y. Plan in [CP10].

2.2. Reconstruction of a cone. Given a subset $\mathcal{S} = \{x_1, \dots, x_s\}$ and a sign sequence $(\varepsilon_1, \dots, \varepsilon_s) \in \{\pm 1\}^s$, Lemma IV.1 shows that if the generalized interpolation problem defined by (i), (ii) and (iii) has a solution then support pursuit recovers exactly all measures σ with support $\mathcal{S}$ and such that $\mathrm{sgn}(\sigma_i) = \varepsilon_i$. Let us emphasize that the result is slightly stronger. Indeed the proof (see page 62) remains unchanged if some coefficients σ_i are zero. Consequently support pursuit recovers exactly all the measures

σ of which support is included in $\mathcal{S} = \{x_1, \ldots, x_s\}$ and such that $\mathrm{sgn}(\sigma_i) = \varepsilon_i$ for all nonzero σ_i. Denote $\mathcal{C}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s)$ this set. It is exactly the cone defined by

$$\mathcal{C}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s) = \Big\{ \sum_{i=1}^{s} \mu_i \, \delta_{x_i} \;\Big|\; \forall \mu_i \neq 0, \quad \mathrm{sgn}(\mu_i) = \varepsilon_i \Big\}.$$

Thus the existence of P implies the exact reconstruction of all measures in this cone. The cone $\mathcal{C}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s)$ is the conic span of an $(s-1)$-dimensional face of the TV-unit ball, that is

$$\mathcal{F}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s) = \Big\{ \sum_{i=1}^{s} \varepsilon_i \lambda_i \, \delta_{x_i} \;\Big|\; \forall\, i,\ \lambda_i \geq 0 \text{ and } \sum_{i=1}^{s} \lambda_i = 1 \Big\}.$$

Furthermore, the affine space $\{\mu,\ \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma)\}$ is tangent to the TV-unit ball at any point $\sigma \in \mathcal{F}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s)$, as shown in the following remark.

Remark — From a convex optimization point of view, the dual certificates [CP10] and the generalized dual polynomials are deeply related: the existence of a generalized dual polynomial P implies that, for all $\sigma \in \mathcal{F}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s)$, a subgradient Φ_P of the TV-norm at the point σ is perpendicular to the set of the feasible points, that is

$$\{\mu,\ \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma)\} \subset \ker(\Phi_P),$$

where ker denotes the nullspace. A proof of this remark can be found at page 63. This fact was previously investigated in Figure 1.2, page 7.

2.3. Remark on Condition (i). Obviously, when $u_k = x^k$ for $k = 0, 1, \ldots, n$, the conditions (ii) and (iii) imply that $n \geq s$ and so condition (i). Nevertheless, this implication is not true for a general set of functions $\{u_0, u_1, \ldots, u_n\}$. Moreover Lemma IV.1 can fail if condition (i) is not satisfied. For example, set $n = 0$ and consider a continuous function u_0 satisfying the two conditions (ii) and (iii). In this case, if the target σ belongs to $\mathcal{F}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s)$ (where $x_1, \ldots, x_s$ and $\varepsilon_1, \ldots, \varepsilon_s$ are given by (ii) and (iii)), then every measure $\mu \in \mathcal{F}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s)$ is a solution of support pursuit given the observation $\mathcal{K}_0(\sigma)$. Indeed,

$$\|\mu\|_{TV} = \int_{-1}^{1} u_0 \, \mathrm{d}\mu = \mathcal{K}_0(\mu),$$

for all $\mu \in \mathcal{F}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s)$. This example shows that the condition (i) is necessary. Reading the proof 6, the conditions (ii) and (iii) ensure that the solutions to support pursuit belong to the cone $\mathcal{C}(x_1, \varepsilon_1, \ldots, x_s, \varepsilon_s)$, whereas the condition (i) gives uniqueness.

2.4. The extremal Jordan type measures. Lemma IV.1 shows that Definition IV.1 is well-founded. As a matter of fact, we have the the following corollary.

Corollary — *Let σ be an extremal Jordan type measure. Then the measure σ is a solution to support pursuit given the observation $\mathcal{K}_n(\sigma)$. Furthermore, if the Vandermonde system given by (i) in Lemma IV.1 has full column rank (where $\mathcal{S} = \{x_1, \ldots, x_s\}$ denotes the support of σ), then the measure σ is the unique solution to support pursuit given the observation $\mathcal{K}_n(\sigma)$.*

This corollary shows that the "extremal Jordan type" notion is appropriate to exact reconstruction using support pursuit. In the next section we focus on nonnegative measure which are extremal Jordan type measure (as mentioned in the introduction).

3. Exact reconstruction of the nonnegative measures

In this section we show that if the underlying family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is a homogeneous M-system (notion defined later on) then the support pursuit recovers exactly all nonnegative discrete measures from the observation of few generalized moments.

3.1. The homogeneous Markov-systems. The *Markov*-systems were introduced in approximation theory [KS66, KN77, BE95]. They deal with the problem of finding the best approximation, in terms of the ℓ_∞-norm, of a given continuous function. We begin with the definition of the *Chebyshev*-systems (the so-called T-system). They can be seen as a natural extension of the algebraic monomials. Thus a finite linear combination of elements of a T-system is called a generalized polynomial.

Definition IV.2 (T-system) — *Denote $\{u_0, u_1, \dots, u_k\}$ a set of continuous real (or complex) functions on $\bar{I}$. This set is a T-system of degree k if and only if every generalized polynomial*

$$P = \sum_{l=0}^{k} a_l u_l\,,$$

where $(a_0, \dots, a_k) \neq (0, \dots, 0)$, has at most k zeros in I.

This definition is equivalent to each of the two following conditions:

- ✦ For all $x_0, x_1, \dots, x_k$ distinct elements of I and all $y_0, y_1, \dots, y_k$ real (or complex) numbers, there exists a unique generalized polynomial P (i.e. P belongs to $\mathrm{Span}\{u_0, u_1, \dots, u_k\}$) such that $P(x_i) = y_i$, for all $i = 0, 1, \dots, k$.
- ✦ For all $x_0, \dots, x_k$ distinct elements of I, the *generalized Vandermonde system*,

$$\begin{pmatrix} u_0(x_0) & u_0(x_1) & \dots & u_0(x_k) \\ u_1(x_0) & u_1(x_1) & \dots & u_1(x_k) \\ \vdots & \vdots & & \vdots \\ u_k(x_0) & u_k(x_1) & \dots & u_k(x_k) \end{pmatrix}$$

  has full rank.

Eventually, we say that the family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is a M-system if and only if it is a T-system of degree k for all $0 \leq k \leq n$. Actually the M-systems are common objects (see [KN77]), we mention some examples below.

Usually, the M-systems are defined on general Hausdorff spaces (see [BEZ94] for instance). For sake of readability, we present examples with different values of I. In each case, our results easily extend to target measures with finite support included in the corresponding I. As usual, if not specified, the set I is assumed to be equal to $[-1, 1]$.

- ✦ **Real polynomials:** The family $\mathcal{F}_p = \{1, x, x^2, \dots\}$ is a M-system. The real polynomials give the standard moments.
- ✦ **Müntz polynomials:** Let $0 < \alpha_1 < \alpha_2 < \cdots$ be any real numbers. The family $\mathcal{F}_m = \{1, x^{\alpha_1}, x^{\alpha_2}, \dots\}$ is a M-system on $I = [0, +\infty)$.
- ✦ **Trigonometric functions:** The family $\mathcal{F}_{\cos} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ is a M-system on $I = [0, 1]$.
- ✦ **Characteristic function:** The family $\mathcal{F}_c = \{1, \exp(\imath\pi x), \exp(\imath 2\pi x), \dots\}$ is a M-system on $I = [-1, 1)$. The moments are the characteristic function of σ at points $k\pi$, $k \in \mathcal{N}$. It yields,

$$c_k(\sigma) = \int_{-1}^{1} \exp(\imath k\pi t)\mathrm{d}\sigma(t) = \varphi_\sigma(k\pi)\,.$$

In this case, the underlying scalar field is $\mathbb{C}$.

✦ **Stieltjes transformation:** The family $\mathcal{F}_s = \{\frac{1}{z_1 - x}, \frac{1}{z_2 - x}, \dots\}$, where none of the z_k's belongs to $[-1, 1]$, is a M-system. The corresponding moments are the Stieltjes transformation $S_\sigma(z_k)$ of σ, namely

$$c_k(\sigma) = \int_{-1}^{1} \frac{\mathrm{d}\sigma(t)}{z_k - t} = S_\sigma(z_k)\,.$$

✦ **Laplace transform:** The family $\mathcal{F}_l = \{1, \exp(-x), \exp(-2x), \dots\}$ is a M-system. The moments are the Laplace transform $\mathcal{L}\sigma$ at the integer points. It holds

$$c_k(\sigma) = \int_{-1}^{1} \exp(-kt)\,\mathrm{d}\sigma(t) = \mathcal{L}\sigma(k)\,.$$

A broad variety of common families can be considered in our framework. The above list is not meant to be exhaustive.

Consider the family $\mathcal{F}_s = \{\frac{1}{z_0 - x}, \frac{1}{z_1 - x}, \dots\}$. Remark that no linear combination of its elements gives the constant function 1. Thus the constant function 1 is not a generalized polynomial of this system. To avoid such case, we introduce the *homogeneous* M-systems. We say that a family $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is a homogeneous M-system if and only if it is a M-system and u_0 is a constant function. In this case, all the constant functions c, with $c \in \mathbb{R}$ (or $\mathbb{C}$), are generalized polynomial. Hence the field $\mathbb{R}$ (or $\mathbb{C}$) is naturally embedded in the generalized polynomial. The adjective homogeneous is named after this comment. From any M-system we can always construct a homogeneous M-system. Indeed, let $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ be a M-system. In particular the family $\mathcal{F}$ is a T-system of order 0. Thus the continuous function u_0 does not vanish in $[-1, 1]$. As a matter of fact the family $\{1, \frac{u_1}{u_0}, \frac{u_2}{u_0}, \dots, \frac{u_n}{u_0}\}$ is a homogeneous M-system.

All the previous examples of M-systems (see 3.1) are homogeneous, even the Stieljes transformation considering:

$$\widetilde{\mathcal{F}}_s = \left\{1, \frac{1}{z_1 - x}, \frac{1}{z_2 - x}, \dots\right\}.$$

Using homogeneous M-systems, We show that one can exactly recover all nonnegative measures from few generalized moments.

3.2. The theorem of exact reconstruction. The following result is one of the main theorem of this chapter. It states that the support pursuit (IV.2) recovers all nonnegative measures σ, of which size of support is s, from only $2s + 1$ generalized moments.

Theorem IV.2 ([dCG11]) — *Let $\mathcal{F}$ be an homogeneous M-system on I. Consider a non-negative measure σ with finite support included in I. Then the measure σ is the unique solution to support pursuit given the observation $\mathcal{K}_n(\sigma)$ whenever n is not less than twice the size of the support of σ.*

Proof. The complete proof can be found at page 64 but some key points from the theory of approximation are presented in the following. For further insights about the Markov systems, we recommend the fruitful books [KN77, KS66] to the reader. □

In addition, this result is sharp in the following sense. Every measure, of which size of support is s, depends on $2s$ parameters (s for its support and s for its weights). Surprisingly, this information can be recovered from only $2s + 1$ of its generalized moments.

Furthermore the program (IV.2) does not use the fact that the target is nonnegative. It recovers σ among all the signed measures with finite support.

An important property of the M-systems is the existence of nonnegative generalized polynomial that vanishes exactly at a prescribed set of points $\{t_1, \ldots, t_m\}$, where $t_i \in I$ for all $i = 1, \ldots, m$. Indeed, define *the index* as

$$\text{Index}(t_1, \ldots, t_m) = \sum_{j=1}^{m} \chi(t_j)\,, \tag{IV.4}$$

where $\chi(t) = 2$ if t belongs to $\mathring{I}$ (the interior of I) and 1 otherwise. The next lemma guarantees the existence of nonnegative generalized polynomials.

Lemma IV.3 (Nonnegative generalized polynomial) — *Consider a M-system $\mathcal{F}$ and points $t_1, \ldots, t_m$ in I. These points are the only zeros of a nonnegative generalized polynomial of degree at most n if and only if* $\text{Index}(t_1, \ldots, t_m) \leq n$.

The reader can find a proof of this lemma in [KN77]. Notice that this lemma holds for all M-systems, however our main theorem needs a homogeneous M-system. If one considers non-homogeneous M-systems then it is possible to give counterexamples that goes against Theorem IV.2 for all $n \geq 2s$. Indeed, we have the next result.

Proposition IV.4 ([dCG11]) — *Let σ be a nonnegative measure supported by s points. Let n be an integer such that $n \geq 2s$. Then there exists a M-system $\mathcal{F}$ and a measure $\mu \in \mathcal{M}$ such that $\mathcal{K}_n(\sigma) = \mathcal{K}_n(\mu)$ and $\|\mu\|_{TV} < \|\sigma\|_{TV}$.*

Proof. See page 64. □

Theorem IV.2 gives us the opportunity to build a large family of deterministic matrices for Compressed Sensing in the case of nonnegative signals.

3.3. Deterministic matrices for Compressed Sensing. The heart of this chapter beats in the next theorem. It gives deterministic matrices for Compressed Sensing. In Section 2 at page 5, the reader may find some *state-of-the-art* results in Compressed Sensing. In particular, if

$$n \geq C\, s \log\left(\frac{p}{s}\right),$$

where $C > 0$ is a universal constant, then there exists (with high probability) a random matrix $A \in \mathbb{R}^{n \times p}$ such that basis pursuit recovers all s-sparse vectors from the observation Ax_0. Considering nonnegative sparse vectors (i.e. vectors with nonnegative entries), we drop the bound on n to

$$n \geq 2s + 1\,. \tag{IV.5}$$

Unlike the above examples, our result holds for all values of the parameters (as soon as $n \geq 2s + 1$). In addition we give explicit design matrices for basis pursuit. Last but not least, the bound on n does not depend on p.

Remark — This result is not new. Actually, it is known that one can construct deterministic "neighborly polytopes" from *moment curves*. Furthermore, D.L. Donoho and J. Tanner [DT05] have shown that one can construct deterministic design, in the range described by (IV.5), for which basis pursuit exactly recovers any s-sparse vector. These design are related to the moment and the trigonometric families in the present framework. Notice that we extend their result to all homogeneous M-systems. I thank J. Tanner for giving me this reference.

Theorem IV.5 ([DT05, dCG11], Deterministic design) — *Let n, p, s be integers such that*

$$s \leq \min(n/2,\, p).$$

Let $\{1, u_1, \ldots, u_n\}$ be a homogeneous M-system on I. Let $t_1, \ldots, t_p$ be distinct reals of I. Let A be the generalized vandermonde system defined by

$$A = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ u_1(t_1) & u_1(t_2) & \ldots & u_1(t_p) \\ u_2(t_1) & u_2(t_2) & \ldots & u_2(t_p) \\ \vdots & \vdots & & \vdots \\ u_n(t_1) & u_n(t_2) & \ldots & u_n(t_p) \end{pmatrix}.$$

Then basis pursuit (I.11) *exactly recovers all nonnegative s-sparse vectors $x_0 \in \mathbb{R}^p$ from the observation Ax_0.*

Proof. See page 65. □

Although the predictors could be highly correlated, basis pursuit exactly recovers the target vector x_0. Of course, this result is theoretical. In actual practice, the sensing matrix A can be very ill-conditioned. In this case, basis pursuit behaves poorly.

We run some numerical experiments that illustrate Theorem IV.5. They are of the following form:

(a) Choose constants s (the sparsity), n (the number of known moments), and p (the length of the vector). Choose the family $\mathcal{F}$ (cosine, polynomial, Laplace, Stieljes,...).

(b) Select the subset $\mathcal{S}$ (of size s) uniformly at random.

(c) Randomly generate an s-sparse vector x_0 of which support is $\mathcal{S}$, and such that its nonzero entries are distributed according to the chi-square distribution with 1 degree of freedom.

(d) Compute the observation Ax_0.

(e) Solve (I.11), and compare to the target vector x_0.

✧ As mentioned in Chapter I, the program (I.11) can be recasted in a linear program. Then we use an interior point method to solve (I.11).

✧ The entries of the target signal are distributed according to the chi-square distribution with 1 degree of freedom. We chose this distribution to ensure that the entries are nonnegative. Let us emphasized that the actual values of x_0 can be arbitrary, only its sign matters. The result remains the same if one take the nonzero entries equal to 1, say.

✧ Let us denote $K : t \mapsto (1, u_1(t), \ldots, u_n(t))$. The columns of A are the values of this map at points $t_1, \ldots, t_p$. For large p, the vectors $K(t_i)$ can be very correlated. As a matter of fact, the matrix A can be ill-conditioned to solve the program (I.11). To avoid such a case, we chose a family such that the map K has a large derivative function. It appears that the cosine family gives very good numerical results (see Figure 4.2).

✧ We investigate the reconstruction error between the numerical result $\tilde{x}$ of the program (I.11) and the target vector x_0. Our experiment is of the following form:

(a) Choose p (the length of the vector) and N (the number of numerical experiments).

(b) Let s such that $1 \leq s \leq (p-1)/2$.

(c) Set $n = 2s + 1$ and solve the program (I.11). Let $\tilde{x}$ be the numerical result.

(d) Compute the ℓ_1-error $\|\tilde{x} - x_0\|_1 / p$.

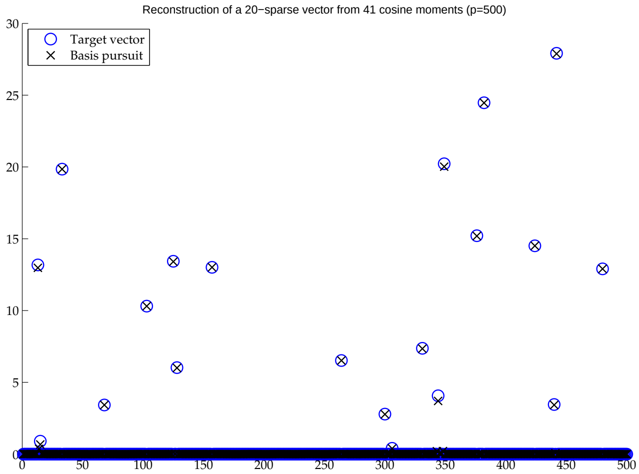


FIGURE 4.2. Consider the family $\mathcal{F}_{\cos} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ on $I = [0,1]$ and the points $t_k = k/501$, for $k = 1, \dots, 500$. The blue circles represent the target vector x_0 (a 20-sparse vector), while the black crosses represent the solution $x^\star$ of (I.11) from the observation of 41 cosine moments. In this example $s = 20$, $n = 41$, and $p = 500$. More numerical results can be found in the Appendix 7.

(e) Repeat N times the steps (c) and (d), and compute Err_s the arithmetic mean of the ℓ_1-errors.

(f) Return $\|\mathrm{Err}_s\|_\infty$, the maximal value of Err_s when $1 \leq s \leq (p-1)/2$.

For $p = 100$ and $N = 10$, we find that

$$\|\mathrm{Err}_s\|_\infty \leq 0.05\,.$$

Remark that all the experiments were done for $n = 2s + 1$. This is the smallest value of n such that Theorem 3.3 holds.

4. Exact reconstruction for generalized Chebyshev measures

In this section we give some examples of extremal polynomials P as they appears in Definition IV.1. Considering M-systems, the corollary of Lemma IV.1 shows that every measure with Jordan support included in (E_P^+, E_P^-) is the only solution to support pursuit. Indeed, the condition (i) of Lemma IV.1 is clearly satisfied when the underlying family $\mathcal{F}$ is a M-system.

4.1. Trigonometric families. In the context of M-system we can exhibit some very particular dual polynomials. The global extrema of these polynomials gives families of support on which results of Lemma IV.1 hold. For instance, consider the $(n+1)$-dimensional cosine system

$$\mathcal{F}_{\cos}^n := \{1, \cos(\pi x), \dots, \cos(n\pi x)\}$$

on $I = [0,1]$. Obviously, the extremal polynomials $P_k(x) = \cos(k\pi x)$, for $k = 1, \dots, n$, satisfy $\|P_k\|_\infty \leq 1$ and $P_k(\ell/k) = (-1)^\ell$, for $\ell = 0, \dots, (k-1)$. Following Definition IV.1, denote

✧ $E^+_{P_k} := \{2\ell/k \mid \ell = 0, \dots, \lfloor \frac{k-1}{2} \rfloor\}$,

✧ $E^-_{P_k} := \{(2\ell-1)/k \mid \ell = 1, \dots, \lfloor \frac{k}{2} \rfloor\}$.

The corollary that follows Lemma IV.1 asserts the following result: Consider a signed measure σ having Jordan support $(\mathcal{S}^+, \mathcal{S}^-)$ such that $\mathcal{S}^+ \subset E^+_{P_k}$ and $\mathcal{S}^- \subset E^-_{P_k}$, for some $1 \leq k \leq n$. Then the measure σ can be exactly reconstructed from the observation of

$$\int_0^1 \cos(k\pi t) \mathrm{d}\sigma(t), \quad k = 0, 1, \dots, n. \tag{IV.6}$$

Since the family $\mathcal{F}^n_{\cos}$ is a M-system, the condition (i) in Lemma IV.1 is satisfied. Hence, the measure σ is the only solution of support pursuit given the observations (IV.6). Moreover, using the classical mapping,

$$\Psi : \begin{cases} [0,1] & \to & [-1,1] \\ x & \mapsto & \cos(\pi x) \end{cases},$$

the system of function $(1, \cos(\pi x), \dots, \cos(n\pi x))$ is pushed forward on the system of function $(1, T_1(x), \dots, T_n(x))$ where $T_k(x)$ is the so-called Chebyshev polynomial of the first kind of order k, $k = 1, \dots, n$ (see 4.2).

By the same way, consider the complex value M-system defined by

$$\mathcal{F}^n_c = \{1, \exp(\imath\pi x), \dots, \exp(\imath n\pi x)\}$$

on $I = [0, 2)$. In this case, one can check that

$$P_{\alpha,k}(t) = \cos(k\pi(t-\alpha)), \quad \forall t \in [0, 2),$$

where $\alpha \in \mathbb{R}$ and $0 \leq k \leq n/2$, is a generalized polynomial. Following the previous example, we set

✧ $E^+_{P_{\alpha,k}} := \{\alpha + 2l/k \pmod 2 \mid l = 0, \dots, \lfloor \frac{k-1}{2} \rfloor\}$,

✧ $E^-_{P_{\alpha,k}} := \{\alpha + (2l-1)/k \pmod 2 \mid l = 1, \dots, \lfloor \frac{k}{2} \rfloor\}$.

Hence Lemma IV.1 can be applied. It yields that any signed measure having Jordan support included in $(E^+_{P_{\alpha,k}}, E^-_{P_{\alpha,k}})$, for some $\alpha \in \mathbb{R}$ and $1 \leq k \leq n/2$, is the unique solution of support pursuit given the observation

$$\int_0^2 \exp(\imath k\pi t) \mathrm{d}\sigma(t) = \varphi_\sigma(k\pi), \quad \forall k = 0, \dots, n,$$

where $\varphi_\sigma(k\pi)$ has been defined in the previous section (see 3.1). Notice that the study of basis pursuit with this kind of trigonometric moments have been considered in the pioneering work of D.L. Donoho and P.B. Stark [DS89].

4.2. The Chebyshev polynomials. As mentioned in the introduction, the k-th Chebyshev polynomial of the first order is defined by

$$T_k(x) = \cos(k \arccos(x)), \quad \forall x \in [-1, 1].$$

We give some well known properties of the Chebyshev polynomials. The k-th Chebyshev polynomial satisfies the *equioscillation property* on $[-1, 1]$. As a matter of fact, there exists $k+1$ points $\zeta_i = \cos(\pi i/k)$ with $1 = \zeta_0 > \zeta_1 > \dots > \zeta_k = -1$ such that

$$T_k(\zeta_i) = (-1)^i \|T_k\|_\infty = (-1)^i,$$

where the supremum norm is taken over $[-1,1]$. We draw the firsts Chebyshev polynomials on Figure 4.3.

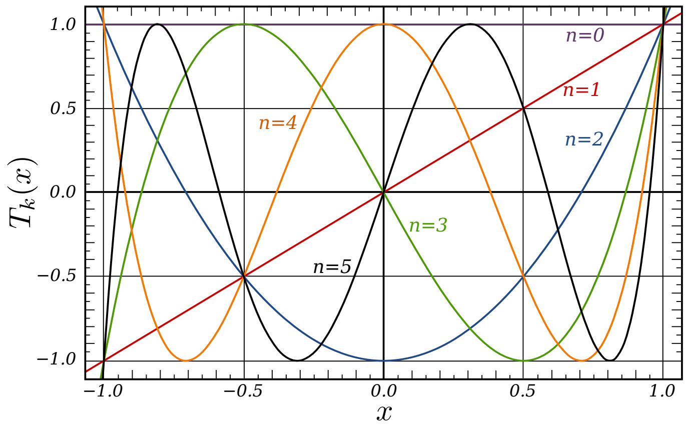


FIGURE 4.3. The firsts Chebyshev polynomials.

Moreover, the Chebyshev polynomial T_k satisfies the following extremal property.

Theorem IV.6 ([Riv90, BE95]) — *We have*

$$\min_{p\in\mathcal{P}^{\mathbb{C}}_{k-1}} \left\|x^k - p(x)\right\|_\infty = \left\|2^{1-k}T_k\right\|_\infty = 2^{1-k}\,,$$

where $\mathcal{P}^{\mathbb{C}}_{k-1}$ denotes the set of the complex polynomials of degree less than $k-1$, and the supremum norm is taken over $[-1,1]$. Moreover, the minimum is uniquely attained by $p(x) = x^k - 2^{1-k}T_k(x)$.

These two properties, namely the equioscillation property and the extremal property, will be useful to us when defining the generalized Chebyshev polynomial. As usual, using Lemma IV.1 we uncover an exact reconstruction result. Consider the family

$$\mathcal{F}^n_p = \{1, x, x^2, \dots, x^n\}$$

on $I = [-1,1]$. Set

- $E^+_{T_k} = \left\{\cos(2l\pi/k),\ l = 0,\dots,\lfloor\frac{k}{2}\rfloor\right\}$,
- $E^-_{T_k} = \left\{\cos((2l+1)\pi/k),\ l = 0,\dots,\lfloor\frac{k}{2}\rfloor\right\}$.

The following result holds: consider a signed measure σ having Jordan support included in $\left(E^+_{T_k}, E^-_{T_k}\right)$, for some $1 \le k \le n$. Then the measure σ is the only solution to support pursuit given its $(n+1)$ first standard moments using support pursuit. As a matter of fact, this result can be extended to any M-systems considering the generalized Chebyshev polynomials.

4.3. The generalized Chebyshev polynomials. Following [BE95], define the generalized Chebyshev polynomials as follows. Let $\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ be a M-system on I. Define the *generalized Chebyshev polynomial*

$$\mathfrak{T}_k := \mathfrak{T}_k\{u_0, u_1, \dots, u_n; I\}\,,$$

where $1 \le k \le n$, is defined by the following three properties:

- $\mathfrak{T}_k$ is a generalized polynomial of degree k, i.e. $\mathfrak{T}_k \in \mathrm{Span}\{u_0, u_1, \dots, u_k\}$,

- there exists an alternation sequence, $x_0 < x_1 < \cdots < x_k$, that is,

$$\operatorname{sgn}(\mathfrak{T}_k(x_{i+1})) = -\operatorname{sgn}(\mathfrak{T}_k(x_i)) = \pm\|\mathfrak{T}_k\|_\infty\,, \tag{IV.7}$$

  for $i = 0, 1, \ldots, k-1$,
- and

$$\|\mathfrak{T}_k\|_\infty = 1 \quad \text{with} \quad \mathfrak{T}_k(\max I) > 0\,. \tag{IV.8}$$

The existence and the uniqueness of such $\mathfrak{T}_k$ is proved in [BE95]. Moreover, the following theorem shows that the extremal property implies the equioscillation property (IV.7).

Theorem IV.7 ([Riv90, BE95]) — *The k-th generalized Chebyshev polynomial $\mathfrak{T}_k$ exists and can be written as*

$$\mathfrak{T}_k = c\left(u_k - \sum_{i=0}^{k-1} a_i u_i\right),$$

where $a_0, a_1, \ldots, a_{k-1} \in \mathbb{R}$ are chosen to minimize

$$\left\| u_k - \sum_{i=0}^{k-1} a_i u_i \right\|_\infty,$$

and the normalization constant $c \in \mathbb{R}$ can be chosen so that $\mathfrak{T}_k$ satisfies property (IV.8).

The generalized Chebyshev polynomials give a new family of extremal Jordan type measure (see Definition IV.1). The corresponding target measures are named the *Chebyshev measures*. Considering the equioscillation property (IV.7), set

- $E^+_{\mathfrak{T}_k}$ as the set of the alternation point x_i such that $\operatorname{sgn}(\mathfrak{T}_k(x_i)) = \|\mathfrak{T}_k\|_\infty$,
- $E^-_{\mathfrak{T}_k}$ as the set of the alternation point x_i such that $\operatorname{sgn}(\mathfrak{T}_k(x_i)) = -\|\mathfrak{T}_k\|_\infty$.

A direct consequence of the last definition is the following proposition.

Proposition IV.8 ([dCG11]) — *Let σ be a signed measure having Jordan support included in $(E^+_{\mathfrak{T}_k}, E^-_{\mathfrak{T}_k})$, for some $1 \leq k \leq n$. Then σ is the unique solution to support pursuit* (IV.2) *given $\mathcal{K}_n(\sigma)$, i.e. its $(n+1)$ first generalized moments.*

In the special case $k = n$, Proposition IV.8 shows that support pursuit recovers all signed measures with Jordan support included in $(E^+_{\mathfrak{T}_n}, E^-_{\mathfrak{T}_n})$ from $(n+1)$ first generalized moments. Remark that $E^+_{\mathfrak{T}_n} \cup E^-_{\mathfrak{T}_n}$ has size n. Hence, this proposition shows that, among all the signed measure on $[-1,1]$, support pursuit can recover a signed measure of which support has size n from only $(n+1)$ generalized moments. As a matter of fact, any measure with Jordan support included in $(E^+_{\mathfrak{T}_n}, E^-_{\mathfrak{T}_n})$ can be uniquely defined by only $(n+1)$ generalized moments.

As far as we know, it is difficult to give the corresponding generalized Chebyshev polynomials for a given family $\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$. Nevertheless, P. Borwein, T. Erdélyi, and J. Zhang [BEZ94] gives the explicit form of $\mathfrak{T}_k$ for the rational spaces (i.e. the Stieljes transformation in our framework). See also [DS89, HSS96] for some applications in optimal design. We consider the case of the Stieljes transformation described in Section 3. In this case, the Chebyshev polynomials $\mathfrak{T}_k$ can be precisely described. Consider the homogeneous M-system on $[-1,1]$ defined by

$$\widetilde{\mathcal{F}}^n_s = \left\{1, \frac{1}{z_1 - x}, \frac{1}{z_2 - x}, \ldots, \frac{1}{z_n - x}\right\},$$

where $(z_i)_{i=1}^k \subset \mathbb{C} \setminus [-1,1]$. Reproducing [BE95], we can construct the generalized Chebyshev polynomials of the first kind. As a matter of fact, it yields

$$\mathfrak{T}_k(x) = \frac{1}{2}(f_k(z) + f_k(z)^{-1}), \quad \forall x \in [-1,1],$$

where z is uniquely defined by $x = \dfrac{1}{2}(z + z^{-1})$ and $|z| < 1$, and f_k is a known analytic function in a neighborhood of the closed unit disk. Moreover this analytic function can be only expressed in terms of $(z_i)_{i=1}^k$. We refer to [BE95] for further details.

5. The nullspace property for the measures

In this section we consider any countable family $\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ of continuous functions on I. In particular we do not assume that $\mathcal{F}$ is a non-homogeneous M-system. We aim at deriving a sufficient condition for exact reconstruction of signed measures. More precisely, we concern with giving a related property to the nullspace property [CDD09] of Compressed Sensing (see DefinitionI.2 at page 7). Remark that the solutions to the program (IV.2) depend only on the $(n+1)$ first elements of $\mathcal{F}$ and on the target measure σ. We investigate the condition that must satisfy the family $\mathcal{F}$ to ensure exact reconstruction. Consider the linear map $\mathcal{K}_n : \mu \mapsto (c_0(\mu), \ldots, c_n(\mu))$ from $\mathcal{M}$ to $\mathbb{R}^{n+1}$. We refer to this map as the *generalized moment morphism*. Its nullspace $\ker(\mathcal{K}_n)$ is a linear subspace of $\mathcal{M}$. The Lebesgue's decomposition theorem is the precious tool that carves the nullspace property. Let $\mu \in \mathcal{M}$ and $S = \{x_1, \ldots, x_s\}$ be a finite subset of I. Define $\Delta_S = \sum_{i=1}^{s} \delta_{x_i}$ as the Dirac comb with support S. The Lebesgue's decomposition of μ with respect to Δ_S gives

$$\mu = \mu_S + \mu_{S^c}, \tag{IV.9}$$

where μ_S is a discrete measure of which support is included in S, and μ_{S^c} is a measure of which support is included in $S^c := I \setminus S$. Define the nullspace property with respect to a Jordan support family Υ.

Definition IV.3 (Nullspace property with respect to a Jordan support family Υ) — *We say that the generalized moment morphism $\mathcal{K}_n$ satisfies the nullspace property with respect to a Jordan support family Υ if and only if it satisfies the following property. For all nonzero measures μ in the nullspace of $\mathcal{K}_n$, and for all $(\mathcal{S}^+, \mathcal{S}^-) \in \Upsilon$,*

$$\|\mu_S\|_{TV} < \|\mu_{S^c}\|_{TV}, \tag{IV.10}$$

where $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$. The weak nullspace property states as follows: For all nonzero measures μ in the nullspace of $\mathcal{K}_n$, and for all $(\mathcal{S}^+, \mathcal{S}^-) \in \Upsilon$,

$$\|\mu_S\|_{TV} \leq \|\mu_{S^c}\|_{TV},$$

where $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$.

Given a nonzero measure μ in the nullspace of $\mathcal{K}_n$, this property means that more than half of the total variation of μ cannot be concentrated on a small subset. The nullspace property is a key to exact reconstruction as shown in the following proposition.

Proposition IV.9 ([dCG11]) — *Let Υ be a Jordan support family. Let σ be a signed measure having a Jordan support in Υ. If the generalized moment morphism $\mathcal{K}_n$ satisfies the nullspace property with respect to Υ. Then, the measure σ is the unique solution of support pursuit* (IV.2) *given the observation $\mathcal{K}_n(\sigma)$. If the generalized moment morphism $\mathcal{K}_n$ satisfies* *the weak nullspace property with respect to Υ. Then, the measure σ is the a solution of support pursuit* (IV.2) *given the observation $\mathcal{K}_n(\sigma)$.*

Proof. See page 65. □

As far as we know, it is difficult to check the nullspace property. In the following, we give an example such that the weak nullspace property is satisfied.

5.1. The spaced out interpolation. We recall that S_Δ is the set of all pairs (S^+, S^-) of subsets of $I = [-1, 1]$ such that

$$\forall x, y \in S^+ \cup S^-,\ x \neq y, \quad |x - y| \geq \Delta. \tag{IV.11}$$

The next lemma shows that if Δ is large enough then there exists a polynomial of degree n, with supremum norm not greater than 1, that interpolates 1 on the set $\mathcal{S}^+$ and -1 on the set $\mathcal{S}^-$.

Lemma IV.10 — *For all $(S^+, S^-) \in S_\Delta$, there exists a polynomial $P_{(S^+,S^-)}$ such that*

✦ *$P_{(S^+,S^-)}$ has degree n not greater than $(2/\sqrt{\pi})\,(\sqrt{e}/\Delta)^{5/2+1/\Delta}$,*
✦ *$P_{(S^+,S^-)}$ is equal to 1 on the set S^+,*
✦ *$P_{(S^+,S^-)}$ is equal to -1 on the set S^-,*
✦ *and $\|P_{(S^+,S^-)}\|_\infty \leq 1$ over I.*

Proof. See page 66. □

This upper bound is meant to show that one can interpolate any sign sequence on S_Δ. Let us emphasize that this result is far from being sharp. Considering L_2-minimizing polynomials under fitting constraint, the authors of the present chapter think that one can greatly improve the upper bound of Lemma IV.10. As a matter of fact, our numerical experiments is in complete agreement with this comment. Invoking Lemma IV.1, Lemma IV.10 gives the next proposition.

Proposition IV.11 — *Let Δ be a positive real. If $n \geq (2/\sqrt{\pi})\,(\sqrt{e}/\Delta)^{5/2+1/\Delta}$ then $\mathcal{K}_n$ satisfies the weak nullspace property with respect to S_Δ.*

Proof. See page 66. □

The bound $(2/\sqrt{\pi})\,(\sqrt{e}/\Delta)^{5/2+1/\Delta}$ can be considerably improved in actual practice. The following numerical experiment shows that this bound can be greatly lowered.

Our numerical experiment consists in looking for a generalized polynomial satisfying the assumption of Lemma IV.1. We work here with the cosine system $(1, \cos(\pi x), \cos(2\pi x), \ldots, \cos(n\pi x))$ for various values of the integer n. As explained in Section 4, we can also push this system on the more classical power system $(1, x, x^2, \ldots, x^n)$. So that, our numerical experiments may be interpreted in this last frame. We consider signed measure having a support $\mathcal{S}$ with $|\mathcal{S}| = 10$. We consider Δ-spaced out type measures for various values of Δ. For each choice of Δ, we draw uniformly 100 realizations of signed measures. This means that the points of $\mathcal{S}$ are uniformly drawn on I^{10}, where $I = [0, 1)$ here, with the restriction that the minimal distance between two points is at least Δ and that there exists a couple of points that are exactly Δ away from each other. Further, we uniformly randomized the signs of the measure on each point of $\mathcal{S}$. As we wish to work with true signed measures, we do not allow the case where all the signs are the same (negative or positive measures). Once we simulated the set $\mathcal{S}^+$ and $\mathcal{S}^-$, we wish to build an interpolating polynomial P of

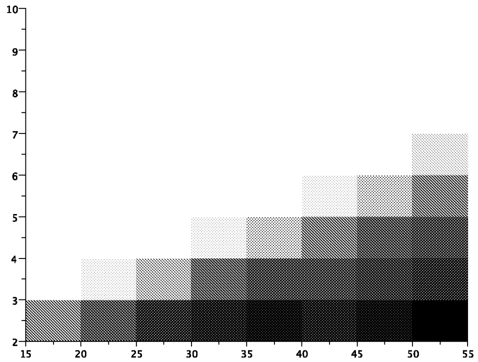


FIGURE 4.4. Consider the family $\mathcal{F}_{\cos} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ on $I = [0,1]$. Set $s = 10$ the size of the target support. We are concerned with signed measures with Jordan support in S_Δ (see (IV.11)). The abscissa represents the values of $1/\Delta$ (with $\Delta = 1/15, 1/20, \dots, 1/55$), and the ordinate represent the values of n (with $n = 20, 30, \dots, 100$). For each value of (Δ, n), we draw uniformly 100 realizations of signed measures and the corresponding L_2-minimizing polynomial P. The gray scale represents the percentage of times that $\|P\|_\infty \leq 1$ occurs. The white color means 100% (support pursuit exactly recovers all the signed measures) while the black color represent 0% (in all our experiments, the polynomial P is such that $\|P\|_\infty > 1$ over I).

degree n having value 1 on $\mathcal{S}^+$, -1 on $\mathcal{S}^-$ and having a supremum norm minimum. As this last minimization is not obvious, we relax it to the minimization of the L_2-norm with the extra restriction that the derivative of the interpolation polynomial vanishes on $\mathcal{S}$. Hence, when this last optimization problem has a solution having a supremum norm not greater that 1 Lemma IV.1 may be applied and support pursuit leads to exact reconstruction. The proportion of experimental results, where the supremum norm of the L_2 optimal polynomial is not greater than 1, is reported in Figure 4.4. In our experiments we consider the values $\Delta = 1/15, 1/20, \dots, 1/55$. According to Proposition IV.11, the corresponding value of n range from 10^{19} to 10^{59}. In our experiments, we find that $n = 80$ suffices.

6. Proofs

Proof of Lemma IV.1 — Assume that a generalized dual polynomial P exists. Let σ be such that $\sigma = \sum_{i=1}^{s} \sigma_i \, \delta_{x_i}$, with $\operatorname{sgn}(\sigma_i) = \varepsilon_i$. Let $\sigma^\star$ be a solution of the support pursuit (IV.2) then $\int P \, \mathrm{d}\sigma = \int P \, \mathrm{d}\sigma^\star$. The equality (ii) yields $\|\sigma\|_{TV} = \int P \, \mathrm{d}\sigma$. Combining the two previous equalities,

$$\|\sigma\|_{TV} = \int P \, \mathrm{d}\sigma = \int P \, \mathrm{d}\sigma^\star = \sum_{i=1}^{s} \varepsilon_i \, \sigma_i^\star + \int P \, \mathrm{d}\sigma^\star_{\mathcal{S}^c} \,,$$

where $\varepsilon_i = \operatorname{sgn}(\sigma_i)$ and

$$\sigma^\star = \sum_{i=1}^{s} \sigma_i^\star \, \delta_{x_i} + \sigma^\star_{\mathcal{S}^c} \,,$$

according to the Lebesgue decomposition (IV.9). Since $\|P\|_\infty = 1$, it holds

$$\sum_{i=1}^{s} \varepsilon_i \sigma_i^\star + \int P \, d\sigma^\star_{\mathcal{S}^c} \leq \|\sigma^\star_{\mathcal{S}}\|_{TV} + \|\sigma^\star_{\mathcal{S}^c}\|_{TV} = \|\sigma^\star\|_{TV} .$$

Observe $\sigma^\star$ is a solution of the support pursuit, it follows that $\|\sigma\|_{TV} = \|\sigma^\star\|_{TV}$ and the above inequality is an equality. It yields $\int P \, d\sigma^\star_{\mathcal{S}^c} = \|\sigma^\star_{\mathcal{S}^c}\|_{TV}$. Moreover we have the following result.

Lemma IV.12 — *Let ν be a measure with its support included in $\mathcal{S}^c$. If $\int P \, d\nu = \|\nu\|_{TV}$ then $\nu = 0$.*

Proof. Consider the compact sets

$$\forall k > 0 , \quad \Omega_k := I \setminus \bigcup_{i=1}^{s} \,]x_i - \tfrac{1}{k}, \, x_i + \tfrac{1}{k}[\,,$$

Suppose that there exists $k > 0$ such that $\|\nu_{\Omega_k}\|_{TV} \neq 0$. Then the inequality (iii) leads to $\int_{\Omega_k} P \, d\nu < \|\nu_{\Omega_k}\|_{TV}$. It yields

$$\|\nu\|_{TV} = \int P \, d\nu = \int_{\Omega_k} P \, d\nu + \int_{\Omega_k^c} P \, d\nu < \|\nu_{\Omega_k}\|_{TV} + \left\|\nu_{\Omega_k^c}\right\|_{TV} = \|\nu\|_{TV} \,,$$

which is a contradiction. We deduce that $\|\nu_{\Omega_k}\|_{TV} = 0$, for all $k > 0$. The equality $\nu = 0$ follows with $\mathcal{S}^c = \cup_{k>0} \Omega_k$. ☐

This lemma shows that $\sigma^\star$ is a discrete measure with its support included in $\mathcal{S}$. In this case, the moment constraint $\mathcal{K}_n(\sigma^\star - \sigma) = 0$ can be written as a generalized Vandermonde system,

$$\begin{pmatrix} u_0(x_1) & u_0(x_2) & \dots & u_0(x_s) \\ u_1(x_1) & u_1(x_2) & \dots & u_1(x_s) \\ \vdots & \vdots & & \vdots \\ u_n(x_1) & u_n(x_2) & \dots & u_n(x_s) \end{pmatrix} \begin{pmatrix} \sigma_1^\star - \sigma_1 \\ \sigma_2^\star - \sigma_2 \\ \vdots \\ \sigma_s^\star - \sigma_s \end{pmatrix} = 0 \,.$$

From the condition (i), we deduce that the above generalized Vandermonde system is injective. This concludes the proof. ☐

Proof of the remark in Section 2 page 51 — Let σ belong to $\mathcal{F}(x_1, \varepsilon_1, \dots, x_s, \varepsilon_s)$. Consider the linear functional,

$$\Phi_f : \mu \mapsto \int_I f \, d\mu,$$

where f denotes a continuous bounded function. By definition, any subgradient Φ_f of the TV-norm at point σ satisfies that, for all measures $\mu \in \mathcal{M}$,

$$\|\mu\|_{TV} - \|\sigma\|_{TV} \geq \Phi_f(\mu - \sigma).$$

So that, one can easily check that f is equal to 1 (resp. -1) on $\mathrm{supp}(\sigma^+)$ (resp. $\mathrm{supp}(\sigma^-)$) and that $\|f\|_\infty = 1$. Conversely, any function f satisfying the latter condition leads to a subgradient Φ_f. Therefore, when it exists, the generalized dual polynomial P is such that Φ_P is a subgradient of the TV-norm at point σ. Furthermore, let μ be a feasible point (i.e. $\mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma)$). Since P a generalized polynomial of order n, we deduce that $\Phi_P(\mu - \sigma) = 0$. Hence, the subgradient Φ_P is perpendicular to the set of the feasible points. $\square$

Proof of Theorem IV.2 — The proof essentially relies on Lemma IV.1. Let s be an integer. Let σ be a nonnegative measure. Let $\mathcal{S} = \{x_1, \ldots, x_s\} \subset I$ be its support. The next lemma shows the existence of a generalized dual polynomial.

Lemma IV.13 (Dual polynomial) **—** *Let s be an integer and n be such that $n = 2s$. Let $\mathcal{F}$ be an homogeneous M-system on I. Let $(x_1, \ldots, x_s)$ be such that* $\mathrm{Index}(x_1, \ldots, x_s) \leq n$*. Then there exists a generalized polynomial P of degree d such that*

- ✧ $s \leq d \leq n$,
- ✧ $P(x_i) = 1,\ \forall\, i = 1, \ldots, s$,
- ✧ *and* $|P(x)| < 1$ *for all* $x \notin \{x_1, \ldots, x_s\}$.

We recall that Index is defined by (IV.4). Notice that these polynomials are presented in the first example of Definition IV.1.

Proof of Lemma IV.13. Let $(x_1, \ldots, x_s)$ be such that $\mathrm{Index}(x_1, \ldots, x_s) \leq n$. From Lemma IV.3, there exists a nonnegative polynomial Q of degree d that vanishes exactly at the points x_i. Moreover, its degree d satisfies (i).

Since Q is continuous on the compact set $\overline{I}$ then it is bounded. There exists a real c such that $\|Q\|_\infty < 1/c$. The generalized polynomial

$$P = 1 - cQ\,,$$

is the expected generalized polynomial. This concludes the proof. $\square$

Observe that

- ✧ Using Lemma IV.13, it yields that there exists a generalized dual polynomial, of degree at most $n = 2s$, which interpolates the value 1 at points $\{x_1, \ldots, x_s\}$.
- ✧ Since $\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ is a T-system, the Vandermonde system given by (i) in Lemma IV.1 has full column rank.

Lemma IV.1 concludes the proof. $\square$

Remark — Since $\mathcal{F}$ is a homogeneous M-system, the constant function 1 is a generalized polynomial. Remark the linear combination $P = 1 - cQ$ is a generalized polynomial because 1 is a generalized polynomial. This assumption is essential (see Proposition IV.4).

Proof of Proposition IV.4 — Let $\sigma = \sum_{i=1}^{s} \sigma_i \delta_{x_i}$ be a nonnegative measure. Denote $\mathcal{S} = \{x_1, \ldots, x_s\}$ its support. Let n be an integer such that $n \geq 2s$.

Step 1: Let $\mathcal{F}_h = \{1, u_1, u_2, \ldots\}$ be an homogeneous M-system (the standard polynomials for instance). Let $t_1, \ldots, t_{n+1} \in I \setminus \mathcal{S}$ be distinct points. It follows that the Vandermonde system $\begin{pmatrix} 1 & \ldots & 1 \\ u_1(t_1) & \ldots & u_1(t_{n+1}) \\ \vdots & & \vdots \\ u_n(t_1) & \ldots & u_n(t_{n+1}) \end{pmatrix}$ has full rank. It yields that we may choose $(\nu_1, \ldots, \nu_{n+1}) \in \mathbb{R}^{n+1}$ such that

- ✧ $\nu = \sum_{i=1}^{n+1} \nu_i \delta_{t_i}$,
- ✧ and for all $k = 0, \ldots, n$, $\int_I u_k \mathrm{d}\nu = \int_I u_k \mathrm{d}\sigma$.

Step 2: Set
$$r=\frac{\|\sigma\|_{TV}}{\|\nu\|_{TV}+1}\,.$$
Consider a positive continuous functions u_0 such that
- $u_0(x_i)=r$, for $i=1,\ldots,s$,
- $u_0(t_i)=1$, for $i=1,\ldots,n+1$,
- the function u_0 is not constant.

Set $\mathcal{F}=\{u_0,u_0\,u_1,u_0\,u_2,\ldots\}$. Obviously, $\mathcal{F}$ is a non-homogeneous M-system. As usual, denote $\mathcal{K}_n$ the generalized moment morphism of order n derived from the family $\mathcal{F}$.

Last step: Set $\mu=r\,\nu$. An easy calculation gives $\mathcal{K}_n(\sigma)=\mathcal{K}_n(\mu)$. Remark that
$$\|\mu\|_{TV}=\sum_{i=1}^{n+1} r\,|\nu_i|=\frac{\sum_{i=1}^{n+1}|\nu_i|}{\sum_{i=1}^{n+1}|\nu_i|+1}\,\|\sigma\|_{TV}<\|\sigma\|_{TV}\,,$$
this concludes the proof. □

Proof of Theorem IV.5 — Set $\mathcal{T}=\{t_1,\ldots,t_p\}$. Let us denote $\mathcal{M}_{\mathcal{T}}$ the set of all finite measure of which support is included in $\mathcal{T}$. Let $\Theta_{\mathcal{T}}$ be the linear map defined by
$$\Theta_{\mathcal{T}}:\left\{\begin{array}{ccc}(\mathbb{R}^p,\ell_1) & \to & (\mathcal{M}_{\mathcal{T}},\|.\|_{TV})\\ (x_1,\ldots,x_p) & \mapsto & \displaystyle\sum_{i=1}^{p} x_i\,\delta_{t_i}\end{array}\right..$$
One can check that $\Theta_{\mathcal{T}}$ is a bijective isometry. Moreover, it holds
$$\forall y\in\mathbb{R}^p,\quad \mathcal{K}_n(\Theta_{\mathcal{T}}(y))=Ay, \tag{IV.12}$$
where A is the generalized vandermonde system defined by
$$A=\begin{pmatrix}1 & 1 & \ldots & 1\\ u_1(t_1) & u_1(t_2) & \ldots & u_1(t_p)\\ u_2(t_1) & u_2(t_2) & \ldots & u_2(t_p)\\ \vdots & \vdots & & \vdots\\ u_n(t_1) & u_n(t_2) & \ldots & u_n(t_p)\end{pmatrix}.$$
In the meantime, let x_0 be a nonnegative s-sparse vector. Set $\sigma=\Theta_{\mathcal{T}}(x_0)$ then the size support of σ is at most s. Consequently, Theorem IV.2 shows that σ is the unique solution to support pursuit. Since $\sigma\in\mathcal{M}_{\mathcal{T}}$, it yields that σ is the unique solution to the following program
$$\sigma=\operatorname*{Arg\,min}_{\mu\in\mathcal{M}_{\mathcal{T}}}\|\mu\|_{TV}\quad \text{s.t. } \mathcal{K}_n(\mu)=\mathcal{K}_n(\sigma)\,.$$
Using (IV.12) and the isometry $\Theta_{\mathcal{T}}$, it gives that x_0 is the unique solution to the following program
$$x_0=\operatorname*{Arg\,min}_{y\in\mathbb{R}^p}\|y\|_1\quad \text{s.t. } Ay=Ax_0\,.$$
This concludes the proof. □

Proof of Proposition IV.9 — Let $\mathcal{K}_n$ be a generalized moment morphism that satisfies the nullspace property with respect to a Jordan support family Υ. Let σ be a signed measure of which Jordan support belongs to Υ. Let $\sigma^\star$ be a solution of the support pursuit (IV.2), it follows that $\|\sigma^\star\|_{TV} \leq \|\sigma\|_{TV}$. Denote $\mu = \sigma^\star - \sigma$ and remark that $\mu \in \ker(\mathcal{K}_n)$. It holds

$$\begin{aligned}\|\sigma^\star\|_{TV} &= \|\sigma^\star_{\mathcal{S}}\|_{TV} + \|\sigma^\star_{\mathcal{S}^c}\|_{TV}\,,\\ &= \|\sigma + \mu_{\mathcal{S}}\|_{TV} + \|\mu_{\mathcal{S}^c}\|_{TV}\,,\\ &\geq \|\sigma\|_{TV} - \|\mu_{\mathcal{S}}\|_{TV} + \|\mu_{\mathcal{S}^c}\|_{TV}\,,\end{aligned}$$

where $\mathcal{S}$ denotes the support of σ. Suppose that $\mu \neq 0$. The nullspace property yields that the measure μ satisfies the inequality (IV.10). We deduce $\|\sigma^\star\|_{TV} > \|\sigma\|_{TV}$ which is a contradiction. Thus $\mu = 0$ and $\sigma^\star = \sigma$. This concludes the proof. □

Proof of Lemma IV.10 — For sake of readability, we present the sketch of the proof here. Let $(S^+, S^-) \in S_\Delta$. Set $S = S^+ \cup S^- = \{x_1, \ldots, x_s\}$. Consider the Lagrange interpolation polynomials

$$\ell_k(x) = \frac{\prod_{i\neq k}(x - x_i)}{\prod_{i\neq k}(x_k - x_i)}\,,$$

for $1 \leq k \leq s$. One can bound the supremum norm of ℓ_k over $[0,1]$ by

$$\|\ell_k\|_\infty \leq L(\Delta),$$

where $L(\Delta)$ is an upper bound that depends only on Δ. Consider the m-th Chebyshev polynomial of the first order $T_m(x) = \cos(m \arccos(x))$, for all $x \in [-1,1]$. For a sufficient large value of m, there exists $2s$ extrema ζ_i of T_m such that $|\zeta_i| \leq 1/(sL(\Delta))$. Interpolating values ζ_i at point x_k, we build the expected polynomial P. We find that the polynomial P has degree not greater than

$$C\,(\sqrt{e}/\Delta)^{5/2+1/\Delta}\,,$$

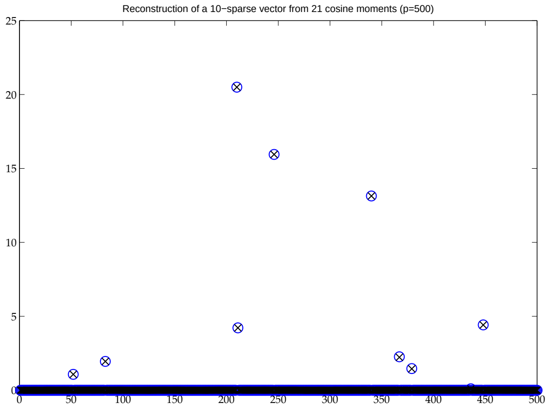
where $C = 2/\sqrt{\pi}$. □

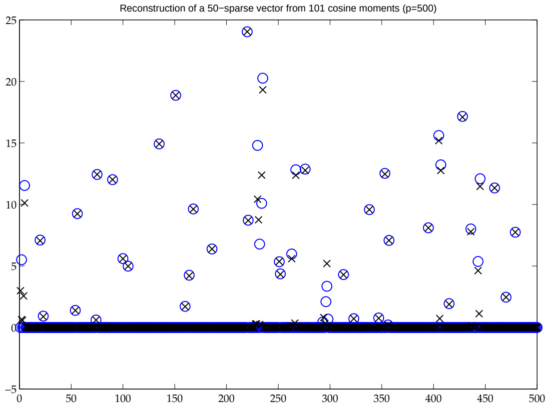
Proof of Proposition IV.11 — Let μ be a nonzero measure in the nullspace of $\mathcal{K}_n$ and $(\mathcal{A}, \mathcal{B})$ be in S_Δ. Let $\mathcal{S}$ be equal to $\mathcal{A} \cup \mathcal{B}$. Set $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) the set of points x in $\mathcal{S}$ such that the μ-weight at point x is nonnegative (resp. negative). Observe that $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$ and $(\mathcal{S}^+, \mathcal{S}^-) \in S_\Delta$. From Lemma IV.10, there exists $P_{(S^+,S^-)}$ of degree not greater than n such that $P_{(S^+,S^-)}$ is equal to 1 on S^+, -1 on S^-, and $\|P_{(S^+,S^-)}\|_\infty \leq 1$. It yields

$$\begin{aligned}\int P_{(S^+,S^-)}\mathrm{d}\mu &= \|\mu_S\|_{TV} + \int_{S^c} P_{(S^+,S^-)}\mathrm{d}\mu\,,\\ &\geq \|\mu_S\|_{TV} - \|\mu_{S^c}\|_{TV}\,.\end{aligned}$$

Since $\mu \in \ker(\mathcal{K}_n)$, it follows that $\int P_{(S^+,S^-)}\mathrm{d}\mu = 0$. This concludes the proof. □

7. Numerical Experiments





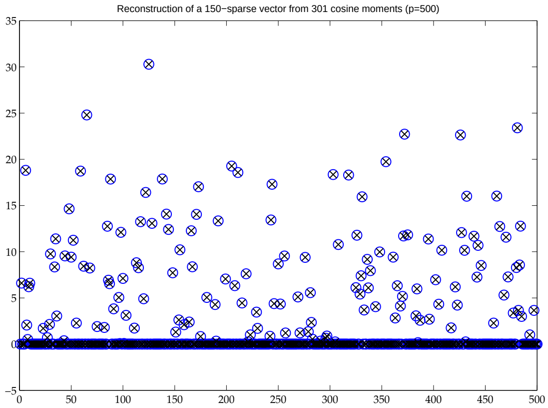


FIGURE 4.5. These numerical experiments illustrate Theorem IV.5. We consider the family $\mathcal{F}_{\cos} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ and the points $t_k = k/(p+1)$, for $k = 1, \dots, p$. The blue circles represent the target vector x_0, while the black crosses represent the solution $x^\star$ of (I.11). The respective values are $s = 10$, $n = 21$, $p = 500$; $s = 50$, $n = 101$, $p = 500$; and $s = 150$, $n = 301$, $p = 500$.

CHAPITRE V

Inégalités isopérimétriques sur la droite réelle

Résumé — Dans un récent papier, A. Cianchi, N. Fusco, F. Maggi, et A. Pratelli ont montré que, dans l'espace de Gauss, un ensemble de mesure donnée et de frontière de Gauss presque minimal est nécessairement proche d'être un demi-espace.

En utilisant uniquement des outils géométriques, nous étendons leur résultat au cas des mesures log-concaves symétriques sur la droite réelle. On donne des inegalités isopérimétriques quantitatives optimales et l'on prouve que parmi les ensembles de mesure donnée et d'asymétrie donnée (distance à la demi-droite, i.e. distance aux ensembles de périmètre minimal), les intervalles ou les complémentaires d'intervalles ont le plus petit périmètre.

Abstract — In a recent paper A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli have shown that, in the Gauss space, a set of given measure and almost minimal Gauss boundary measure is necessarily close to be a half-space.

Using only geometric tools, we extend their result to all symmetric log-concave measures on the real line. We give sharp quantitative isoperimetric inequalities and prove that among sets of given measure and given asymmetry (distance to half line, i.e. distance to sets of minimal perimeter), the intervals or complements of intervals have minimal perimeter.

1. Quantitative isoperimetric inequalities

In this part, we study the Gaussian isoperimetric inequality in dimension $n = 1$ and we prove a sharp quantitative version of it. More precisely, denote the one-dimensional Gaussian measure by

$$\gamma := \exp(-t^2/2)/\sqrt{2\pi} \cdot \mathcal{L}^1,$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure. The classical Gaussian isoperimetric inequality [CS74] states that among sets of given measure in $(\mathbb{R}^n, \gamma^n)$, where γ^n denotes the standard n-dimensional Gaussian measure, half spaces have minimal Gauss boundary measure. This reads as

$$P_{\gamma^n}(\Omega) \geq J_\gamma\left(\gamma^n(\Omega)\right),$$

where J_γ is optimal (and defined later on in the text). In their paper [CFMP11] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli have derived an improvement of the form

$$P_{\gamma^n}(\Omega) - J_\gamma(\gamma^n(\Omega)) \geq \Theta_{\gamma^n}\left(\gamma^n(\Omega), \lambda(\Omega)\right) \geq 0,$$

where $\lambda(\Omega)$ measures (in a suitable sense, see formula (V.7) below) how far Ω is from a half-space, and Θ_{γ^n} is a function of two variables, whose form depends on the reference measure γ^n, and such that $\Theta_{\gamma^n}(x, y) \to 0$ as $y \to 0$, i.e. it tends to zero as $\lambda(\Omega)$ tends to zero (at least for the case of the Gaussian measure). In their result the dependence on

$\lambda(\Omega)$ is precise, whereas the dependence on $\gamma^n(\Omega)$ is not explicit. We focus on the one dimensional case: in this setting Theorem 1.2 of [CFMP11] gives that

$$P_\gamma(\Omega) \geq J_\gamma(\gamma(\Omega)) + \frac{\lambda(\Omega)}{C(\gamma(\Omega))}\sqrt{\log\left(1/\lambda(\Omega)\right)}, \tag{V.1}$$

where $C(\gamma(\Omega))$ is a constant that depends only on $\gamma(\Omega)$. In this chapter, Theorem V.9 is a version of this statement which is actually valid for all symmetric log-concave measures μ on the real line. In addition, when the measure μ is not exponential-like (see Section 3), this quantitative inequality implies that a set of given measure and almost minimal perimeter is necessarily "close" to be a half-line, i.e. an isoperimetric set.

2. The isoperimetric inequality on the real line

In this section, we recall the standard isoperimetric inequality for the log-concave measures, and the definition of the asymmetry which measures the gap between a given set and the sets of minimal perimeter. Let $\mu = f \cdot \mathcal{L}^1$ be a measure with density function f with respect to the 1-dimensional Lebesgue measure. Throughout this chapter, we assume that

(i) f is supported and positive over some interval (a_f, b_f), where a_f and b_f can be infinite,

(ii) μ is a probability measure: $\displaystyle\int_{\mathbb{R}} f = 1$,

(iii) μ is a log-concave measure: By a theorem of Borell [Bor75b], it is equivalent to

$$\forall\, x, y \in (a_f, b_f), \quad \forall\, \theta \in (0,1), \quad f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta},$$

(iv) and μ is symmetric with respect to the origin:

$$\forall\, x \in \mathbb{R}, \quad f(x) = f(-x).$$

Remark — Observe that every measure $\mu(. + \alpha)$, where $\alpha \in \mathbb{R}$, shares the same isoperimetric properties as the measure μ.

We recall the definition of the μ-perimeter. Denote by Ω a measurable set. Following [Fed69], define the set Ω^d of all points with density exactly $d \in [0,1]$ as

$$\Omega^d = \left\{ x \in \mathbb{R}, \quad \lim_{\rho \to 0} \frac{\mathcal{L}^1(\Omega \cap B_\rho(x))}{\mathcal{L}^1(B_\rho(x))} = d \right\},$$

where $B_\rho(x)$ denotes the ball with center x and radius ρ. Define the essential boundary $\partial^M\Omega$ as the set $\mathbb{R} \setminus \left(\Omega^0 \cup \Omega^1\right)$. Define the μ-perimeter as

$$P_\mu(\Omega) = \mathcal{H}^0_\mu(\partial^M\Omega) = \int_{\partial^M\Omega} f(x)\,\mathrm{d}\mathcal{H}^0(x), \tag{V.2}$$

where $\mathcal{H}^0$ is the Hausdorff measure of dimension 0 over $\mathbb{R}$ and $\mathcal{H}^0_\mu := f \cdot \mathcal{H}^0$. The isoperimetric function I_μ of the measure μ is defined by

$$I_\mu(r) = \inf_{\mu(\Omega)=r} P_\mu(\Omega). \tag{V.3}$$

Under Assumption (iii), we can give an explicit form to the isoperimetric function using a so-called function J_μ. Indeed, denote F the cumulative distribution function of the measure μ. Since the function f is supported and positive over some interval (a_f, b_f) then the cumulative distribution function is increasing on the interval (a_f, b_f). Define

$$J_\mu(r) = f\big(F^{-1}(r)\big), \tag{V.4}$$

where r is such that $0 < r < 1$, $J_\mu(0) = J_\mu(1) = 0$, and F^{-1} denotes the inverse function of F. Following the article [Bob94] of S.G. Bobkov, since the measure μ is symmetric with respect to the origin, then the inverse function of F satisfies,

$$F^{-1}(r) = \int_{1/2}^{r} \frac{\mathrm{d}t}{J_\mu(t)}, \quad \forall r \in (0,1). \tag{V.5}$$

Using (V.5), one can check [Bob94] that the following lemma holds.

Lemma V.1 — *The measure μ is log-concave if and only if J_μ is concave on $(0,1)$.*

Furthermore, it is known [Bor75a] that the infima of (V.3) are exactly (up to a μ-negligible set) the intervals $(-\infty, \sigma_-)$ and $(\sigma_+, +\infty)$, where $\sigma_- = F^{-1}(r)$ and $\sigma_+ = F^{-1}(1-r)$. The isoperimetric inequality states

$$P_\mu(\Omega) \geq J_\mu(\mu(\Omega)), \tag{V.6}$$

where Ω is a Lebesgue measurable set. This shows that, in the log-concave case, the isoperimetric function coincides with the function J_μ. We concern with quantifying the difference between any measurable set Ω and an isoperimetric infimum (i.e. measurable set such that the isoperimetric inequality (V.6) is an equality). Following [CFMP11], define the **asymmetry** $\lambda(\Omega)$ of a set Ω as

$$\lambda(\Omega) = \min\left\{\mu(\Omega \Delta (-\infty, \sigma_-))\,,\ \mu(\Omega \Delta (\sigma_+, +\infty))\right\}, \tag{V.7}$$

where $\sigma_- = F^{-1}(\mu(\Omega))$ and $\sigma_+ = F^{-1}(1-\mu(\Omega))$, and Δ is the symmetric difference operator.

Remark — The name asymmetry [FMP08] is inherited from the case of the Lebesgue measure on $\mathbb{R}^n$. In this case, the sets with minimal perimeter are balls, hence very symmetric.

Define the **isoperimetric projection** of a set Ω as the open half-line achieving the minimum in (V.7). In the case where this minimum is not unique we can choose whatever infima as an isoperimetric projection.

3. Sharp quantitative isoperimetric inequalities

This section gives a sharp improvement of (V.6) involving the asymmetry $\lambda(\Omega)$. In [CFMP11], the authors use a technical lemma (Lemma 4.7, Continuity Lemma) to complete their proof. Their lemma applies in the n-dimensional case and is based on a compactness argument derived from powerful results in geometric measure theory. In the one-dimensional case, our approach is purely geometric and does not involve the continuity lemma.

3.1. The shifting lemma. The shifting lemma plays a key role in our proof. This lemma was introduced in [CFMP11] for the Gaussian measure. It naturally extends to even log-concave probability measures. For sake of readability, we begin with the shifting property.

Definition V.1 (The shifting property) — *We say that a measure ν satisfies the shifting property when for every open interval (a,b), the following is true:*

- *If $a+b \geq 0$ then for every (a',b') such that $a \leq a' < b' \leq +\infty$ and $\nu((a,b)) = \nu((a',b'))$, it holds $P_\nu((a,b)) \geq P_\nu((a',b'))$. In other words, if an interval is more to the right of 0, shifting it to the right with fixed measure, does not increase the perimeter.*
- *If $a+b \leq 0$ then for every (a',b') such that $-\infty \leq a' < b' \leq b$ and $\nu((a,b)) = \nu((a',b'))$, it holds $P_\nu((a,b)) \geq P_\nu((a',b'))$. In other words, if an interval is more to the left of 0, shifting it to the left with fixed measure, does not increase the perimeter.*

Remark — As the perimeter is complement-invariant, we may also shift "holes". The shifting property is equivalent to the following property.

- If $a+b \geq 0$ then for every (a',b') such that $a \leq a' < b' \leq +\infty$ and $\nu((a,b)) = \nu((a',b'))$, it holds $P_\nu((-\infty,a)\cup(b,+\infty)) \geq P_\nu((-\infty,a')\cup(b',+\infty))$.
- If $a+b \leq 0$ then for every (a',b') such that $-\infty \leq a' < b' \leq b$ and $\nu((a,b)) = \nu((a',b'))$, it holds $P_\nu((-\infty,a)\cup(b,+\infty)) \geq P_\nu((-\infty,a')\cup(b',+\infty))$.

Roughly, the next lemma shows that, for all measures such that Assumptions (i), (ii), and (iv) hold, Assumption (iii) is equivalent to the shifting property.

Lemma V.2 (The shifting lemma) — *Every log-concave probability measure symmetric with respect to the origin has the shifting property.*

Conversely, let f be a continuous function, positive on an open interval and null outside. If the probability measure with density function f is symmetric with respect to the origin and enjoys the shifting property then it is log-concave.

Proof. Let x, r be in $(0,1)$ and t be in $(r/2,\ 1-r/2)$. Define $\varphi(t) = J_\mu(t-r/2) + J_\mu(t+r/2)$. It represents the μ-perimeter of $(F^{-1}(t-r/2), F^{-1}(t+r/2))$ with measure equal to r. The function J_μ is symmetric with respect to $1/2$ since the density function f is supposed to be symmetric. As the function J_μ is concave and symmetric with respect to $1/2$, so is the function φ. In particular φ is non-decreasing on $(r/2, 1/2]$ and non-increasing on $[1/2, 1-r/2)$. This gives the shifting property.
Conversely, let f be a continuous function, positive on an open interval and null outside. Define the isoperimetric function J_μ as in (V.4). We recall that μ is log-concave if and only if J_μ is concave on $(0,1)$. Since the function J_μ is continuous, it is sufficient to have $J_\mu(x) \geq (1/2)\left(J_\mu(x-d) + J_\mu(x+d)\right)$, for all $x \in (0,1)$, where d is small enough to get $x-d \in (0,1)$ and $x+d \in (0,1)$. Let x and d be as in the previous equality. Since μ is symmetric, assume that $x \leq 1/2$. Put $a = F^{-1}(x)$, $b = F^{-1}(1-x)$, $a' = F^{-1}(x+d)$, $b' = F^{-1}(1-x+d)$, then (a',b') is a shift to the right of (a,b). By the shifting property, we get $P_\mu((a,b)) \geq P_\mu((a',b'))$. The function J_μ is symmetric with respect to $1/2$, it yields (see Figure 5.1),

$$\begin{array}{rclcl} P_\mu((a,b)) & = & J_\mu(x) + J_\mu(1-x) & = & 2J_\mu(x), \\ P_\mu((a',b')) & = & J_\mu(x+d) + J_\mu(1-x+d) & = & J_\mu(x+d) + J_\mu(x-d). \end{array}$$

This ends the proof. □

The key idea of the previous lemma is based on standard properties of the concave functions. Nevertheless, it is the main tool to derive quantitative isoperimetric inequalities. We see that the "shifting property" is particular to the one dimensional case and do not extend to higher dimensions.

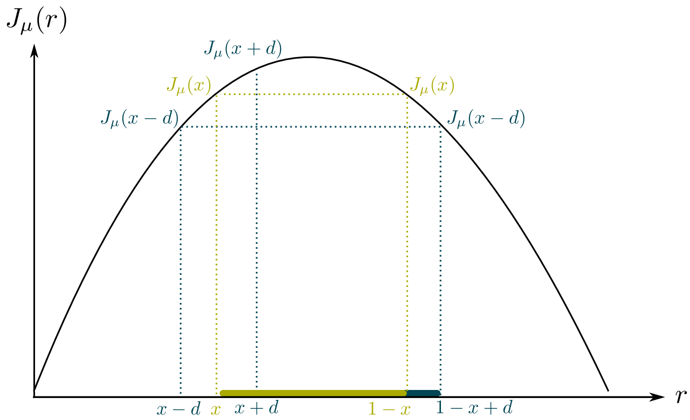


FIGURE 5.1. The log-concavity is equivalent to the shifting property

3.2. Lower bounds on the perimeter. We now recall a result on the structure of sets with finite perimeter on the real line.

Lemma V.3 — *Let Ω be a set of finite μ-perimeter. Then*

$$\Omega = \Big(\bigcup_{n\in I}(a_n,b_n)\Big)\bigcup\mathcal{E},$$

where I is at most countable, $\mathcal{E}$ such that $\mu(\mathcal{E})=0$, and (a_n,b_n) such that

$$d\Big((a_n,b_n),\bigcup_{k\in I\setminus\{n\}}(a_k,b_k)\Big)>0, \tag{V.8}$$

for all n in I, where d denotes the minimal distance between two sets of $\mathbb{R}$.

PROOF. Consider $(K_k)_{k\in\mathbb{N}}$ a sequence of compact sets such that, for all $k\geq 0$, $K_0\subset \ldots \subset K_k\subset(-a_f,a_f)$ and $\cup_{k\in\mathbb{N}}K_k=(-a_f,a_f)$. Then, it yields

$$\Omega=\Big(\bigcup_{k\in\mathbb{N}}(\Omega\cap K_k)\Big)\bigcup E, \tag{V.9}$$

where E is such that $\mu(E)=0$. Let k be an integer. On the compact K_k the function f is bounded from below by a positive real. Thus if $\Omega\cap K_k$ has finite μ-perimeter, it also has finite perimeter. As mentioned in [**AFP00**, **Fed69**], one knows that every set with finite Lebesgue perimeter can be written as at most countable union of open intervals and a set of measure equal to zero. It holds

$$\Omega\cap K_k=\Big(\bigcup_{n\in I_k}(a_n,b_n)\Big)\bigcup\mathcal{E}_k,$$

where I_k is at most countable, $\mathcal{E}_k$ is such that $\mu(\mathcal{E}_k)=0$, and (a_n,b_n) is such that

$$d\Big((a_n,b_n),\bigcup_{l\in I_k\setminus\{n\}}(a_l,b_l)\Big)>0, \tag{V.10}$$

for all n in I_k and d the euclidean distance over the real line. Denote $\mathbb{1}_\Omega$ the indicator function of Ω and $\mathbb{1}'_\Omega$ its distributional derivative. The property (V.10) is a consequence of the fact that $\mathbb{1}'_\Omega$ is locally finite (see [**Fed69**] for instance). Since K_k is compact, the set I_k is finite. One can check that the decomposition (V.9) gives the result. $\square$

The newt lemma shows that among sets of given measure and given asymmetry, the intervals or complements of intervals have minimal perimeter.

Lemma V.4 — *Let Ω be a measurable set with μ-measure at most $1/2$ and $\lambda(\Omega)$ be the asymmetry of Ω. Then, it holds*

$$P_\mu(\Omega) \geq \min\{P_\mu\left(\Omega_c\right), P_\mu\left(\Omega_d\right)\},$$

where

- ✦ $\Omega_c = \big(F^{-1}(\frac{\lambda(\Omega)}{2}),\ F^{-1}(\mu(\Omega) + \frac{\lambda(\Omega)}{2})\big)$,
- ✦ $\Omega_d = \big(-\infty,\ F^{-1}(\mu(\Omega) - \frac{\lambda(\Omega)}{2})\big) \bigcup \big(F^{-1}(1 - \frac{\lambda(\Omega)}{2}),\ +\infty\big)$,

are sets such that $\lambda(\Omega_c) = \lambda(\Omega_d) = \lambda(\Omega)$ and $\mu(\Omega_c) = \mu(\Omega_d) = \mu(\Omega)$.

Let us emphasize that Ω_c and Ω_d have fixed isoperimetric projection (i.e. $(-\infty, -\sigma)$), asymmetry, and measure. Observe that these properties are satisfied only for particular values of $\mu(\Omega)$ and $\lambda(\Omega)$.

Proof. As mentioned in Lemma V.3, assume that $\Omega = \bigcup_{n\in I}(a_n, b_n)$ where I is an at most countable set and (V.8) holds. Suppose that

- ✧ an isoperimetric projection of Ω is $(-\infty, \sigma_-)$ (using a symmetry with respect to the origin if necessary),
- ✧ and that the measure of Ω is at most $1/2$ (and we will see at the end of this section how to extend our result to larger measures).

Then the real number $\sigma_- = F^{-1}(\mu(\Omega))$ is non-positive. Denote $\sigma = -\sigma_-$. Since $\mathbb{1}'_\Omega$ is locally finite, there exists a finite number of sets (a_n, b_n) included in $(-\sigma, \sigma)$, it follows that

$$\Omega = \Big(\bigcup_{h\in\Lambda_-} A_h\Big) \cup I \cup \Big(\bigcup_{h=1}^{N_-} A'_h\Big) \cup \Big(\bigcup_{h=1}^{N_+} B'_h\Big) \cup J \cup \Big(\bigcup_{h\in\Lambda_+} B_h\Big),$$

where

- ✧ Λ_- and Λ_+ are at most countable sets;
- ✧ $A_h = (\alpha_{A_h}, \beta_{A_h})$ with $\beta_{A_h} \leq -\sigma$ (α_{A_h} can be infinite);
- ✧ I is either empty or of the form $I = (\alpha_I, \beta_I)$ with $\alpha_I \leq -\sigma < \beta_I$;
- ✧ A'_h is either empty or of the form $A'_h = (\alpha_{A'_h}, \beta_{A'_h})$ with $-\sigma < \alpha_{A'_h}$ and $\alpha_{A'_h} + \beta_{A'_h} < 0$;
- ✧ B'_h is either empty or of the form $B'_h = (\alpha_{B'_h}, \beta_{B'_h})$ with $\beta_{B'_h} < \sigma$ and $\alpha_{B'_h} + \beta_{B'_h} \geq 0$;
- ✧ J is either empty or of the form $J = (\alpha_J, \beta_J)$ with $\alpha_J < \sigma \leq \beta_J$;
- ✧ and B_h is either empty or of the form $B_h = (\alpha_{B_h}, \beta_{B_h})$ with $\alpha_{B_h} \geq \sigma$ (β_{B_h} can be infinite).

From Ω we build Ω_0 with same measure, same asymmetry, same isoperimetric projection, and lower or equal perimeter. Denote $L = \bigcup_{h\in\Lambda_-} A_h$ and $A_0 = (-\infty, \beta_{A_0})$ where $\beta_{A_0} = F^{-1}(\mu(L))$. Since $\mu(L) \leq \mu(\Omega)$, then $\beta_{A_0} \leq -\sigma$. Using the isoperimetric inequality (V.6) with L, it follows that $P_\mu(A_0) \leq P_\mu(L)$. The same reason gives that there exist a real number $\alpha_{B_0} \geq \sigma$ and a set $B_0 = (\alpha_{B_0}, +\infty)$ with lower or equal perimeter than $\cup_{h\in\Lambda_+} B_h$ (if non-empty). Shift to the left the intervals A'_h until they reach I or $-\sigma$. Shift to the right the intervals B'_h until they reach J or σ. The above operation did not change the amount of mass on left of $-\sigma$ and on the right of σ. We build a set Ω_0 with same asymmetry and same isoperimetric projection as Ω and lower or equal perimeter,

$$\Omega_0 = A_0 \cup I_0 \cup J_0 \cup B_0,$$

where

- $A_0 = (-\infty, \beta_0)$ with $\beta_{A_0} \le -\sigma$;
- I_0 is either empty or of the form $I_0 = (\alpha_{I_0}, \beta_{I_0})$ with $\alpha_{I_0} \le -\sigma < \beta_{I_0}$;
- J_0 is either empty or of the form $J_0 = (\alpha_{J_0}, \beta_{J_0})$ with $\alpha_{J_0} < \sigma \le \beta_{J_0}$;
- and B_0 is either empty or of the form $B_0 = (\alpha_{B_0}, +\infty)$ with $\alpha_{B_0} > \sigma$.

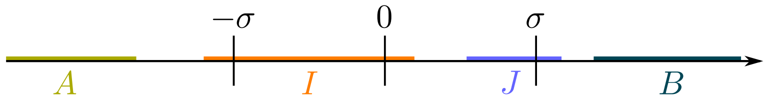


FIGURE 5.2. The set Ω_0

A case analysis on the non-emptiness of sets I_0 and J_0 is required to obtain the claimed result. Every step described below lowers the perimeter (thanks to the shifting lemma, Lemma V.2) and preserves the asymmetry. Before exposing this, we recall that the set Ω_0 is supposed to have $(-\infty, -\sigma)$ as an isoperimetric projection. Thus we pay attention to the fact that it is totally equivalent to ask either the asymmetry to be preserved or the quantity $\lambda(\Omega_0)/2 = \mu(\Omega_0 \cap (-\infty, -\sigma))$ to be preserved trough all steps described below.

- **If I_0 and J_0 are both nonempty:** Applying a symmetry with respect to the origin if necessary, assume that the center of mass of the hole between I_0 and J_0 is not less than 0. We can shift this hole to the right until it touches σ. Using the isoperimetric inequality (V.6), assume that there exist only one interval of the form $(\alpha'_{B_0}, +\infty)$ on the right of σ. We get the case where I_0 is nonempty and J_0 is empty.
- **If I_0 is nonempty and J_0 is empty:** Then shift the hole between A_0 and I_0 to the left until $-\infty$ (there exists one and only one hole between A_0 and I_0 since Ω_0 is not a full measure set of $(-\infty, -\sigma)$). We shift the hole between I_0 and B_0 to the right until $+\infty$ (one readily checks that its center of mass is greater than 0). We get the only interval with same asymmetry and same isoperimetric projection as the set Ω_0. This interval is of the form (the letter c stands for connected),

$$\Omega_c := \left(F^{-1}\left(\lambda(\Omega_0)/2\right),\ F^{-1}\left(\mu(\Omega_0) + \lambda(\Omega_0)/2\right)\right). \tag{V.11}$$

- **If J_0 is nonempty and I_0 is empty:** Shift to the right the hole between J_0 and B_0 to $+\infty$ (there exists one hole between J_0 and B_0 since Ω_0 is not a full measure set of $(\sigma, +\infty)$). We obtain a set $A_0 \cup J'$ where J' is a neighborhood of σ.
  - If $\mu(J') > \mu(A_0)$, then shift J' to the right (which has center of mass greater than 0) till $J' \cap (\sigma, +\infty)$ has weight equal to $\mu(A_0)$ (in order to preserve asymmetry). Using a reflection in respect to the origin, we find ourselves in the case where I_0 is nonempty and J_0 is empty.
  - If $\mu(J') \le \mu(A_0)$, then shift J' (which has center of mass greater than 0) to the right until $+\infty$ and get the case where I_0 and J_0 are both empty.

✦ **If I_0 and J_0 are both empty:** Then the set Ω_0 is of the form (d stands for disconnected),

$$\Omega_d = \big(-\infty, F^{-1}\left(\mu(\Omega_0) - \lambda(\Omega_0)/2\right)\big) \cup \big(F^{-1}\left(1 - \lambda(\Omega_0)/2\right), +\infty\big).$$

This concludes the proof. □

In the following, we describe the conditions on $(\mu(\Omega, \lambda(\Omega))$ for which the sets Ω_c and Ω_d exist. The next lemma shows that asymmetry and perimeter are complement invariant.

Lemma V.5 — *The symmetric difference, the asymmetry, and the perimeter are complement-invariants. Moreover, it holds $m(A) = m(A^c)$ where*

$$m(A) = \min\{\mu(A),\ 1 - \mu(A)\}.$$

Proof. Remark that $\mathbb{1}_{A\Delta B} = |\mathbb{1}_A - \mathbb{1}_B|$, it follows that the symmetric difference is complement-invariant. The essential boundary is complement-invariant, thus Definition V.2 shows that the μ-perimeter is complement-invariant. Considering the symmetry of the isoperimetric function J_μ, we claim that the isoperimetric projections are complements of the isoperimetric projections of the complement. This latter property and the fact that the symmetric difference is complement-invariant give that the asymmetry is complement-invariant. The last equality is easy to check since μ is a probability measure. □

Consider the domain $D = \{(\mu(\Pi), \lambda(\Pi)),\ \Pi \text{ measurable set}\}$. Since the asymmetry is complement-invariant, the domain D is symmetric with respect to the axis $x = 1/2$. Furthermore, we have the next lemma.

Lemma V.6 — *It holds $0 \leq \lambda(\Pi) \leq \min\left(2\, m(\Pi)\,,\, 1 - m(\Pi)\right)$, where Π is a measurable set, and $m(\Pi) = \min\{\mu(\Pi)\,,\, 1 - \mu(\Pi)\}$.*

Proof. Let Π be a measurable set. As asymmetry $\lambda(\Pi)$ and $m(\Pi)$ are complement-invariant (see Lemma V.5), suppose that $\mu(\Pi) \leq 1/2$ thus $m(\Pi) = \mu(\Pi)$. Using symmetry with respect to the origin, suppose that $(-\infty, -\sigma)$ is an isoperimetric projection of Π (where $\sigma = -F^{-1}(\mu(\Pi))$.
We begin with the inequality $\lambda(\Pi) \leq 1 - \mu(\Pi)$. Since $(-\infty, -\sigma)$ is an isoperimetric projection of Π, it holds

$$\mu(\Pi \cap (\sigma, +\infty)) \leq \mu(\Pi \cap (-\infty, -\sigma)) = \mu(\Pi) - \lambda(\Pi)/2.$$

Remark that $\mu((-\sigma, \sigma)) = 1 - 2\,\mu(\Pi)$. Hence, $\lambda(\Pi)/2 = \mu(\Pi \cap (-\sigma, +\infty)) \leq 1 - 2\,\mu(\Pi) + \mu(\Pi) - \lambda(\Pi)/2$, which gives the expected result.
The inequality $\lambda(\Pi) \leq 2\,\mu(\Pi)$ can be deduced from

$$\lambda(\Pi)/2 = \mu((-\infty, -\sigma) \setminus \Pi) \text{ and } \mu((-\infty, -\sigma) \setminus \Pi) \leq \mu((-\infty, -\sigma)) = \mu(\Pi).$$

It is clear that $\lambda(\Pi) \geq 0$, this ends the proof. □

Lemma V.7 — *Let Ω be a measurable set with μ-measure at most $1/2$ and $\lambda(\Omega)$ be the asymmetry of Ω. Then*

✦ *the connected set of the form*

$$\Omega_c = \big(F^{-1}(\lambda(\Omega)/2)\,,\, F^{-1}(\mu(\Omega) + \lambda(\Omega)/2)\big)$$

satisfies $\mu(\Omega_c) = \mu(\Omega)$ and $\lambda(\Omega_c) = \lambda(\Omega)$ when $0 < \lambda(\Omega) \leq 1 - \mu(\Omega)$,

✦ *and the disconnected set of the form*

$$\Omega_d = \big(-\infty\,,\; F^{-1}\big(\mu(\Omega) - \lambda(\Omega)/2\big)\big) \cup \big(F^{-1}\big(1 - \lambda(\Omega)/2\big)\,,\; +\infty\big)$$

satisfies $\mu(\Omega_d) = \mu(\Omega)$ *and* $\lambda(\Omega_d) = \lambda(\Omega)$ *when* $0 < \lambda(\Omega) \le \mu(\Omega)$.

Besides, when $0 < \lambda(\Omega) \le \mu(\Omega)$, $P_\mu(\Omega_d) \le P_\mu(\Omega_c)$ *with equality if and only if* $\mu(\Omega) = 1/2$.

Proof. By construction (see Lemma V.4), the sets Ω_c and Ω_d verify three properties:

✧ their measure is $\mu(\Omega)$,

✧ their asymmetry is $\lambda(\Omega)$,

✧ their isoperimetric projection is $(-\infty, -\sigma)$.

We recall that $\mu(\Omega) \le 1/2$. Using Lemma V.6, it is easy to check that Ω_c satisfies these properties if and only if

$$0 \le \lambda(\Omega) \le \min(2\,\mu(\Omega),\, 1 - \mu(\Omega)). \tag{V.12}$$

Using the definition of the isoperimetric projection, one can check that Ω_d satisfies these properties if and only if

$$0 \le \lambda(\Omega) \le \mu(\Omega). \tag{V.13}$$

Notice that on domain $0 \le \lambda(\Omega) \le \mu(\Omega)$ both sets exist. On this domain,

$$P_\mu(\Omega_d) - P_\mu(\Omega_c) = J_\mu\big(\mu(\Omega) - \lambda(\Omega)/2\big) - J_\mu\big(\mu(\Omega) + \lambda(\Omega)/2\big).$$

Since $\mu(\Omega) - \lambda(\Omega)/2 \le \mu(\Omega) + \lambda(\Omega)/2 \le 1 - \mu(\Omega) + \lambda(\Omega)/2$, we deduce from the concavity and the symmetry of the isoperimetric function that $P_\mu(\Omega_d) \le P_\mu(\Omega_c)$ with equality if and only if $\mu(\Omega) = 1/2$. Using (V.12) and (V.13), we conclude the proof. □

We are concerned with an upper bound on the asymmetry of sets of given measure and given perimeter. Define the **isoperimetric deficit** of Ω as

$$\delta_\mu(\Omega) = P_\mu(\Omega) - J_\mu(\mu(\Omega)). \tag{V.14}$$

Define the **isoperimetric deficit function** K_μ as follows (see Figure 5.3).

✦ On $0 < y \le x \le 1/2$, set $K_\mu(x, y) = J_\mu\,(x - y/2) - J_\mu\,(x) + J_\mu\,(y/2)$.

✦ On $0 < x \le 1/2$ and $x < y \le \min(2x,\, 1 - x)$, set

$$K_\mu(x, y) = J_\mu\,(x + y/2) - J_\mu\,(x) + J_\mu\,(y/2)\,.$$

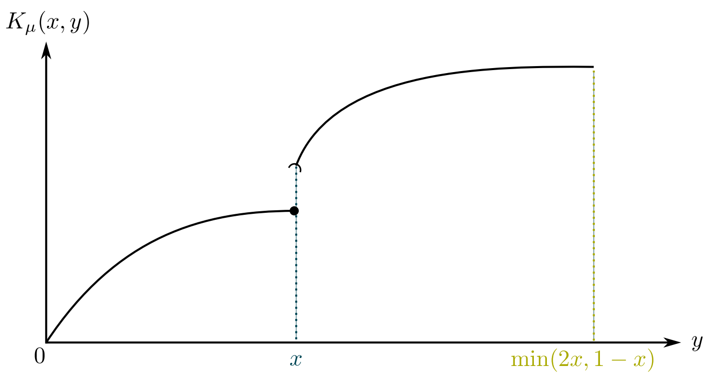


Figure 5.3. The function K_μ is a lower bound on the isoperimetric deficit.

The isoperimetric deficit function $K_\mu(x,y)$ is defined on the domain of all the possible values of $(m(\Omega),\lambda(\Omega))$ (see Lemma V.6). The next lemma focuses on the variations of K_μ.

Lemma V.8 — *Let $0 < x \leq 1/2$. The function $y \mapsto K_\mu(x, y)$ is a non-decreasing lower semi-continuous function. Besides, it is concave on $x < y \leq \min(2x, 1-x)$.*

Proof. The proof is essentially based on the concavity of J_μ.

- **On** $0 < y \leq x$**:** Set

$$\Psi(t) = 1/2\left(J_\mu\left(x/2 - t\right) + J_\mu\left(x/2 + t\right)\right),$$

then the point $(x/2,\, \Psi(t))$ is the middle of the chord joining $(x/2 - t,\, J_\mu(x/2 - t))$ and $(x/2 + t,\, J_\mu(x/2 + t))$. Since J_μ is concave, it is well known that Ψ is a non-increasing function. Remark that $K_\mu(x,y) = 2\,\Psi(x/2 - y/2) - J_\mu\left(x\right)$, thus $y \mapsto K_\mu(x, y)$ is non-decreasing. Moreover the function is continuous as sum of continuous functions.

- **On** $x < y \leq \min(2x, 1-x)$**:** The function $y \mapsto K_\mu(x, y)$ is clearly concave as sum of two concave functions (thus continuous). On this domain,

$$(y/2) + (x + y/2) = x + y \leq x + \min(2x,\, 1-x) \leq 1.$$

Hence the interval $\omega_y = (F^{-1}(y/2), F^{-1}(x + y/2))$ is on the left of the origin. Remark that $K_\mu(x,y) = P_\mu(\omega_y) - J_\mu\left(x\right)$. The shifting lemma (Lemma V.2) applies here and shows that the function $y \mapsto K_\mu(x, y)$ is non-decreasing (as y increases, ω_y shifts to the right).

The variation at x is given by $K_\mu\left(x,x^+\right) - K_\mu(x,x) = J_\mu\left(3/2\,x\right) - J_\mu\left(x/2\right)$, where $K_\mu\left(x,x^+\right) = \lim_{y\to x^+} K_\mu\left(x,y\right)$. One can check that $|1/2 - x/2| \geq |1/2 - 3x/2|$. Using the symmetry with respect to $1/2$ and the concavity of J_μ, one can check that $J_\mu\left(3/2\,x\right) \geq J_\mu\left(x/2\right)$. Hence $K_\mu\left(x,x^+\right) \geq K_\mu(x,x)$.
This discussion shows that $y \mapsto K_\mu(x, y)$ is non-decreasing and lower semi-continuous on the whole domain. This ends the proof. ☐

Define the generalized inverse function of $y \mapsto K_\mu(x, y)$ as

$$K^{-1}_{\mu,x}(d) = \sup\left\{y \mid 0 \leq y \leq \min(2x,\, 1-x) \text{ and } K_\mu(x,y) \leq d\right\}.$$

Lemma V.8 shows that $y \mapsto K_\mu(x, y)$ is a non-decreasing lower semi-continuous function. It is easy to check that $K^{-1}_{\mu,x}$ is non-decreasing. The next theorem is the main result of this chapter.

Theorem V.9 ([dC11a, CFMP11]) **—** *Let Ω be a measurable set and $\lambda(\Omega)$ be the asymmetry of Ω. Set $m(\Omega) = \min\left\{\mu(\Omega)\,,\ 1 - \mu(\Omega)\right\}$, then*

$$\delta_\mu(\Omega) \geq K_\mu(m(\Omega),\, \lambda(\Omega))\,, \tag{V.15}$$

and this inequality is sharp. Moreover, it holds

$$\lambda(\Omega) \leq K^{-1}_{\mu,m(\Omega)}(\delta(\Omega)). \tag{V.16}$$

Proof. Let Ω be a measurable set. If Ω has infinite μ-perimeter the result is true, hence assume that Ω has finite μ-perimeter. Then, it suffices to prove that

- If $0 < \lambda(\Omega) \leq m(\Omega)$ then

$$P_\mu(\Omega) \geq J_\mu\big(m(\Omega) - \lambda(\Omega)/2\big) + J_\mu\big(\lambda(\Omega)/2\big), \tag{V.17}$$

✧ If $m(\Omega) < \lambda(\Omega) \leq \min(2\,m(\Omega), 1 - m(\Omega))$ then

$$P_\mu(\Omega) \geq J_\mu(m(\Omega) + \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2), \tag{V.18}$$

and that these inequalities are sharp. We distinguish four cases (see Figure 5.4).

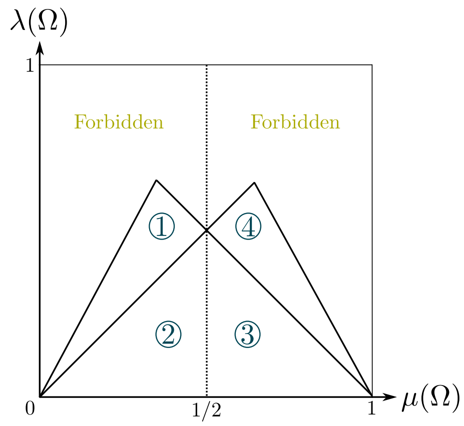


Figure 5.4. Domains of the sets with minimal perimeter given measure and asymmetry

If Ω has measure at most $1/2$, then $m(\Omega) = \mu(\Omega)$. Consider sets Ω_c defined in (V.11) and Ω_d defined in (3.2), compute

$$\begin{aligned} P_\mu(\Omega_d) &= J_\mu(\mu(\Omega) - \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2), \\ P_\mu(\Omega_c) &= J_\mu(\mu(\Omega) + \lambda(\Omega)/2) + J_\mu(\lambda(\Omega)/2). \end{aligned} \tag{V.19}$$

Lemma V.4 says that Ω has μ-perimeter greater or equal than Ω_c or Ω_d.

✦ **Domain 1:** If $\mu(\Omega) < \lambda(\Omega) \leq 1 - \mu(\Omega)$ (and thus $m(\Omega) < \lambda(\Omega) \leq 1 - m(\Omega)$) then from Lemma V.7 we know that Ω_d does not exist for such range of asymmetry. Necessary, it follows that $P_\mu(\Omega) \geq P_\mu(\Omega_c)$. Using (V.19), we complete (V.18).

✦ **Domain 2:** If $0 < \lambda(\Omega) \leq \mu(\Omega)$ (and thus $0 < \lambda(\Omega) \leq m(\Omega)$) then from Lemma V.7 we know that $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$. Thus $P_\mu(\Omega) \geq P_\mu(\Omega_d)$. Using (V.19), we get (V.17).

If Ω has measure greater than $1/2$, then $1 - \mu(\Omega) = m(\Omega)$. The Lemma V.5 shows how to deal with sets of large measure and allows us to consider either Ω or its complement.

✦ **Domain 3:** If $0 < \lambda(\Omega) \leq 1 - \mu(\Omega)$ (and thus $0 < \lambda(\Omega) \leq m(\Omega)$), the complement of Ω satisfies $0 < \lambda(\Omega^c) \leq \mu(\Omega^c)$ (Domain 2). Thus we know that $P_\mu(\Omega_d) \leq P_\mu(\Omega_c)$ (see the previous case on Domain 2). Finally, $P_\mu(\Omega) \geq P_\mu(\Omega_d^c)$ where Ω_d has same asymmetry and measure equal to $m(\Omega)$. Using (V.19), we complete (V.17).

✦ **Domain 4:** If $1 - \mu(\Omega) < \lambda(\Omega) \leq \mu(\Omega)$ (and thus $m(\Omega) < \lambda(\Omega) \leq 1 - m(\Omega)$), the complement of Ω satisfies $\mu(\Omega^c) < \lambda(\Omega^c) \leq 1 - \mu(\Omega^c)$ (Domain 1). From the case on Domain 1, we know that $P_\mu(\Omega^c) \geq P_\mu(\Omega_c)$. Thus, $P_\mu(\Omega) \geq P_\mu(\Omega_c^c)$ where Ω_c has same asymmetry and measure equal to $m(\Omega)$. Using (V.19), we get (V.18).

This case analysis shows (V.15). Set $x = m(\Omega)$, the upper bound (V.16) is a consequence of the definition of $K_{\mu,x}^{-1}$ and (V.15). This concludes the proof. □

Remark — We focus on the Gaussian measure γ. Observe that

$$J_\gamma(t) \underset{t\to 0}{\sim} t\sqrt{2\log(1/t)},$$

so that

$$K_\gamma(x,y) \underset{y\to 0}{\sim} J_\gamma\left(\frac{y}{2}\right) \underset{y\to 0}{\sim} \frac{y}{2}\sqrt{2\log(2/y)}.$$

In particular, there exists a constant $C(x)$ that depends only on x such that

$$K_\gamma(x,y) \geq \frac{y}{C(x)}\sqrt{\log(1/y)}, \quad \text{with } 0 \leq y \leq \min(2x,\, 1-x).$$

Eventually, we recover (V.1) from Theorem V.9.

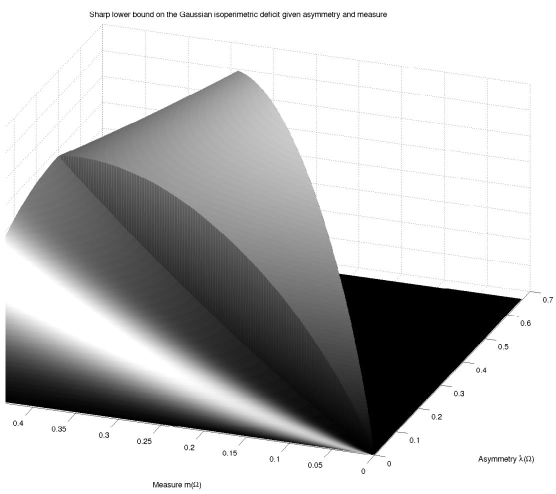


FIGURE 5.5. The function K_γ is a lower bound on the Gaussian isoperimetric deficit.

The equalities (V.19) and the case analysis of the proof of Theorem V.9 give the explicit lower bounds on μ-perimeter.

Proposition V.10 ([dC11a]) **—** *Given two positive numbers μ, λ, we consider the following penalized isoperimetric problem:*

$$\min\{P_\mu(\Omega) \ :\ \Omega \subseteq \mathbb{R},\ \text{with } \mu(\Omega)=\mu \text{ and } \lambda(\Omega)=\lambda\}. \tag{V.20}$$

Then the solution is given by the following sets (see Figure 5.4)

- ✦ $\Omega_c = \left(F^{-1}(\frac{\lambda}{2}),\ F^{-1}(\mu+\frac{\lambda}{2})\right)$, *with* $0<\mu<\lambda\leq 1-\mu$ *and* $\mu\leq 1/2$ *(Domain 1),*
- ✦ $\Omega_d = \left(-\infty,\ F^{-1}(\mu-\frac{\lambda}{2})\right)\cup\left(F^{-1}(1-\frac{\lambda}{2}),\ +\infty\right)$, *with* $0<\lambda\leq\mu$ *and* $\mu\leq 1/2$ *(Domain 2),*
- ✦ $\Omega_d^c = \left(F^{-1}(1-\mu-\frac{\lambda}{2}),\ F^{-1}(1-\frac{\lambda}{2})\right)$, *with* $0<\lambda\leq 1-\mu$ *and* $1/2\leq\mu<1$ *(Domain 3),*
- ✦ $\Omega_c^c = \left(-\infty,\ F^{-1}(\frac{\lambda}{2})\right)\bigcup\left(F^{-1}(1-\mu+\frac{\lambda}{2}),\ +\infty\right)$, *with* $1-\mu<\lambda\leq\mu$ *and* $1/2\leq\mu<1$ *(Domain 4).*

4. Stability of isoperimetric sets

In general the quantitative estimate of Theorem V.9 is not a stability result, since one can have $\delta_\mu = 0$ and $\lambda > 0$ for suitable choices of μ. Indeed, consider the exponential case where $\mu = f \cdot \mathcal{L}^1$ with

$$f(t) = \frac{1}{2}\exp(-|t|), \quad \forall t \in \mathbb{R}.$$

It holds that

$$J_{\exp}(t) = t\,\mathbb{1}_{[0,1/2]} + (1-t)\,\mathbb{1}_{[1/2,1]}.$$

It yields that $K_{\exp} = 0$ on $0 \le y \le x \le 1/2$. Hence, there exists sets with a positive asymmetry and an isoperimetric deficit null. In the case of *the exponential-like distributions* (defined later on), the intervals $(-\infty, F^{-1}(r))$ and $(F^{-1}(1-r), +\infty)$ are not the only sets with minimal perimeter (up to a set of measure equals to 0) given measure r.

We specify this thought defining a natural hypothesis ($\mathbb{H}$). Furthermore, we prove that the asymmetry goes to zero as the isoperimetric deficit goes to zero under ($\mathbb{H}$).

4.1. The hypothesis : non-exponential profile. We can get a better estimate on the asymmetry making another hypothesis. From now, suppose that the measure μ is such that

$$\exists\,\varepsilon > 0 \quad \text{s.t.} \quad t \mapsto J_\mu(t)/t \text{ is decreasing on } (0,\varepsilon). \tag{$\mathbb{H}$}$$

This hypothesis means that J_μ is non-linear in a neighborhood of the origin. We can be more specific introducing the property:

$$\exists\,\varepsilon > 0 \text{ and } c > 0 \quad \text{s.t.} \quad J_\mu(t) = c\,t,\ \forall\, t \in [0,\varepsilon]. \tag{$\overline{\mathbb{H}}$}$$

Since $t \mapsto J_\mu(t)/t$ is non-increasing, it is not difficult to check that ($\overline{\mathbb{H}}$) is the alternative hypothesis of ($\mathbb{H}$). Furthermore, **exponential-like** measure can be defined by the following property:

$$\exists\,\tau > 0 \text{ and } c, c' > 0 \text{ s.t. } f(t) = c'\exp(ct),\ \forall\, t \in (-\infty,\tau). \tag{$\mathcal{E}xp$}$$

Proposition V.11 — *The property ($\overline{\mathbb{H}}$) is equivalent to the property ($\mathcal{E}xp$).*

Proof. The proof is derived from the equality $(F^{-1})'(t) = 1/J_\mu(t)$, for all $t \in (0,1)$ (see [Bob94]). Suppose that the measure satisfies ($\overline{\mathbb{H}}$). Using the above equality for sufficiently small values of r, one can check that $F^{-1}(r) = \dfrac{1}{c}\log(r) + c''$, where c'' is a constant. Hence $F(x) = \exp(c(x-c'')) = \dfrac{c'}{c}\exp(cx)$, which gives the property ($\mathcal{E}xp$). Conversely, suppose that the measure satisfies ($\mathcal{E}xp$). A simple computation gives the property ($\overline{\mathbb{H}}$). □

Suppose that μ satisfies ($\overline{\mathbb{H}}$). It is not difficult to check that the sets (and their symmetric) $(-\infty, F^{-1}(r-s)) \cup (F^{-1}(1-s), +\infty)$, for all $s \in (0,r)$, have minimal perimeter among all sets of given measure r such that $r \le \varepsilon$. It would be natural to define the asymmetry with these sets.

4.2. The continuity theorem. In the following, we give a more convenient bound on the asymmetry. Define the function L_μ as follows.

✦ On $0 < y \le x \le 1/2$, set

$$L_\mu(x,y) = J_\mu(y/2) - y/(2x)\,J_\mu(x)\,.$$

✦ On $0 < x \leq 1/2$ and $x < y \leq \min(2x,\, 1-x)$, set

$$L_\mu(x,\,y) = J_\mu\left(y/2\right) - y/(2(1-x))\, J_\mu\left(x\right).$$

We have the following lemma:

Lemma V.12 — *Let Ω be a measurable set and $\lambda(\Omega)$ be the asymmetry of Ω. Let $m(\Omega) = \min\left\{\mu(\Omega)\,,\, 1-\mu(\Omega)\right\}$ and $\delta_\mu(\Omega) = P_\mu(\Omega) - J_\mu(\mu(\Omega))$. It holds,*

$$\delta_\mu(\Omega) \geq L_\mu(m(\Omega),\, \lambda(\Omega)) \geq 0\,. \tag{V.21}$$

Proof. Since the asymmetry, the perimeter, the isoperimetric deficit, and $m(\Omega)$ are complement invariant, suppose that $m(\Omega) = \mu(\Omega) \leq 1/2$. Set $x = m(\Omega)$ and $y = \lambda(\Omega)$.

✧ **On** $0 < y \leq x$**:** Set $t = y/(2x-y)$ then $x - y/2 = t\,y/2 + (1-t)\,x$. Since J_μ is concave, it holds

$$\begin{aligned} K_\mu(x,y) &= J_\mu\left(x - \frac{y}{2}\right) - J_\mu\left(x\right) + J_\mu\left(\frac{y}{2}\right), \\ &\geq (1+t)J_\mu\left(\frac{y}{2}\right) - tJ_\mu\left(x\right), \\ &= \frac{1}{1-y/2x}\left(J_\mu\left(\frac{y}{2}\right) - \frac{y}{2x}\,J_\mu\left(x\right)\right), \\ &\geq J_\mu\left(\frac{y}{2}\right) - \frac{y}{2x}\,J_\mu\left(x\right). \end{aligned}$$

As J_μ is concave, the function $t \mapsto J_\mu(t)/t$ is non-increasing and we have $(2/y)J_\mu(y/2) - (1/x)J_\mu\left(x\right) \geq 0$.

✧ **On** $x < y \leq \min(2x,\, 1-x)$**:** Using symmetry with respect to $1/2$, remark that

$$\begin{aligned} K_\mu(x,y) &= J_\mu\left(x + \frac{y}{2}\right) - J_\mu\left(x\right) + J_\mu\left(\frac{y}{2}\right) \\ &= J_\mu\left((1-x) - \frac{y}{2}\right) - J_\mu\left(1-x\right) + J_\mu\left(\frac{y}{2}\right) \end{aligned}$$

Substituting x with $1-x$, the same calculus as above can be done.

This ends the proof. □

The lower bound given in Lemma V.12 is the key tool of the proof of the continuity theorem. The hypothesis ($\mathbb{H}$) ensures that the distribution is non-exponential. It is the right framework dealing with continuity as shown in the next theorem.

Theorem V.13 ([dC11a], Continuity for non-exponential distributions) **—** *Assume that the measure μ satisfies the assumption $\mathbb{H}$, then the asymmetry goes to zero as the isoperimetric deficit goes to zero.*

Proof. The proof is based on Lemma V.12 and Theorem V.9. Let $u, v \in (0,1)$, define $\rho(u,v) = J_\mu(u)/u - J_\mu(v)/v$. Suppose $u < v$. Since J_μ is concave, it is easy to check that if $\rho(u,v) = 0$, then $\forall\, u' \leq u$, $\rho(u',v) = 0$. In particular $\mathbb{H}$ implies that $\forall\, u < v\,,\ \rho(u,v) > 0$, for sufficiently small values of v. Remark that $L_\mu(x,y) = (y/2)\rho(y/2,x)$ if $0 < y \leq x$, and $L_\mu(x,y) = (y/2)\rho(y/2, 1-x)$ if $x < y \leq \min(2x,\, 1-x)$. Hence $\mathbb{H}$ implies that $L_\mu > 0$. Using Lemma V.12, it yields that $K_\mu > 0$.

Finally, it is easy to check that if $K_\mu > 0$ then there exists a neighborhood of 0 such that K_μ is increasing. Taking a sufficiently small neighborhood if necessary, one can suppose that K_μ is continuous (the only point of discontinuity of K_μ is $y = x$). On this neighborhood, $K_{\mu,x}^{-1}$ is a continuous increasing function. Using (V.16), this gives the expected result. □

Roughly, a set of given measure and almost minimal boundary measure is necessarily close to be a half-line. Moreover we recover the following well-known result.

Corollary — *Assume that the measure μ satisfies the assumption $\mathbb{H}$, then the half-lines are the unique sets of given measure and minimal perimeter (up to a set of μ-measure null).*

This last results ensure that the asymmetry (V.7) is the relevant notion speaking of the isoperimetric deficit under ($\mathbb{H}$).

As already said, the main argument of our result (Lemma V.2) is peculiar of dimension 1. Nevertheless, it would be interesting to know whether one can extend the results of [CFMP11] to non-exponential log-concave measures also in higher dimensions or not.

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Index